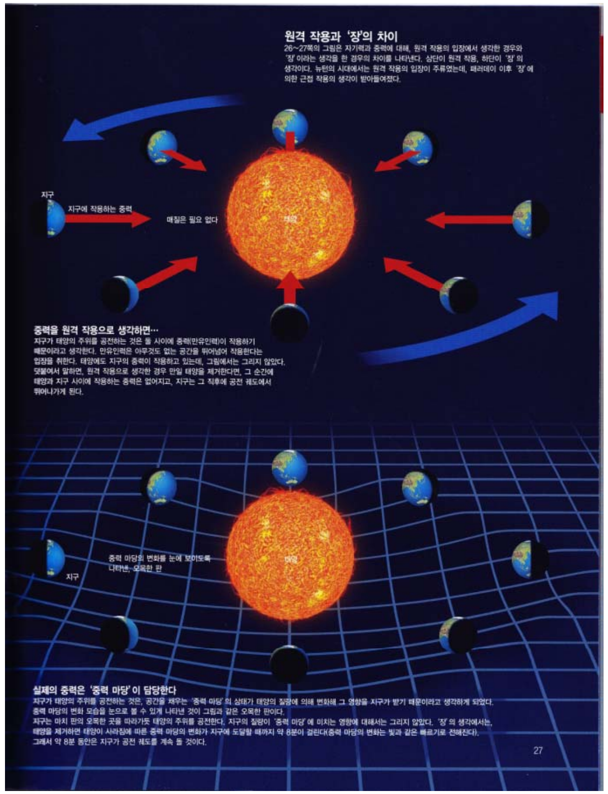
Quantum Electrodynamics

Syllabus

- Introduction (Chap. 1)
- Special Relativity (Chap. 2)
- Quantum Mechanics (Chap. 3)
  - Hydrogen Atom
  - Meson wave function
  - Baryon wave function
  - Magnetic moments
- Detector
- Data Processing
- Feynman diagram (Chap. 4)
- QED (Chap. 5) & Weak interaction
- QCD (Chap. 6)

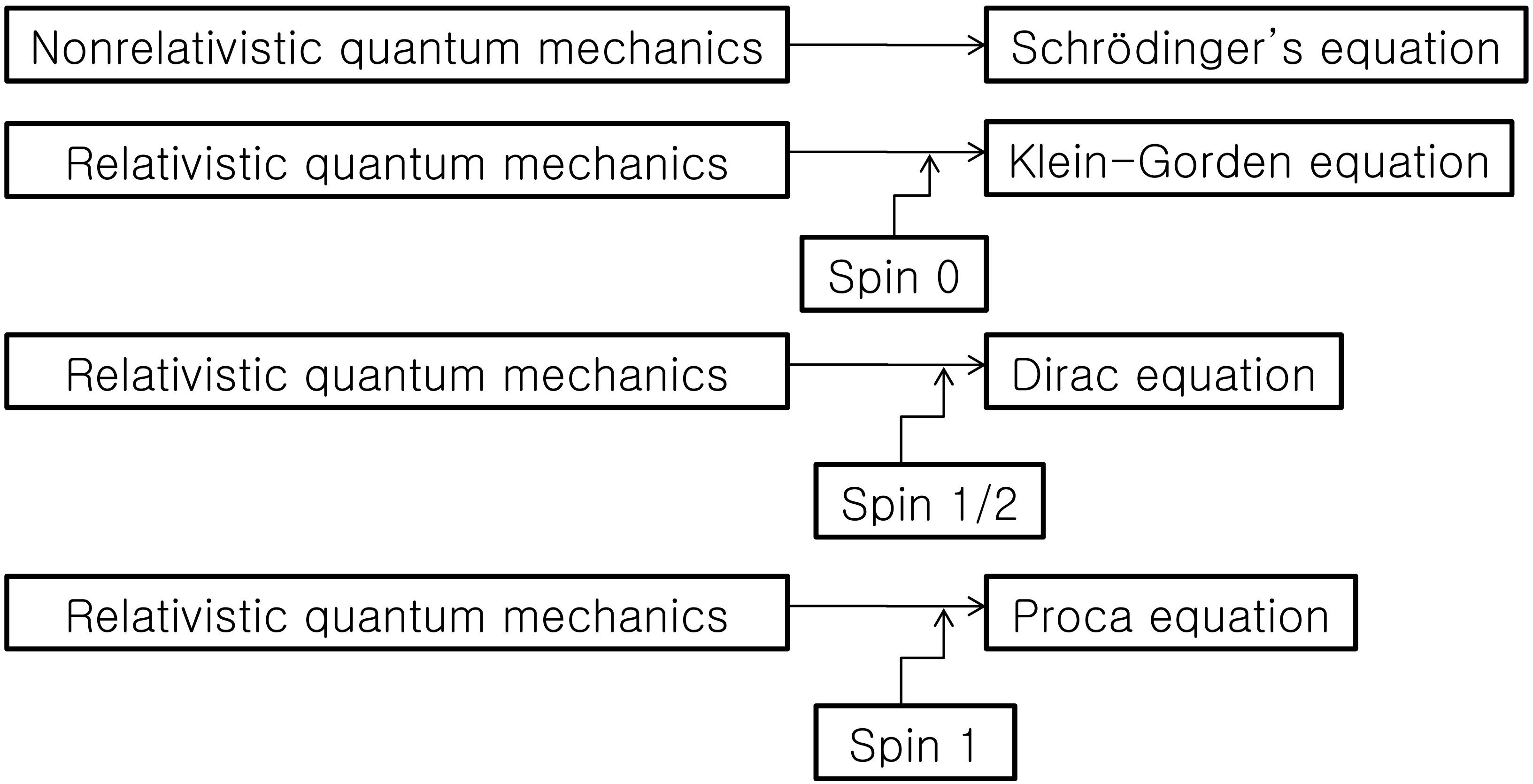
In this chapter....

- Dirac equation.
- Feynman rules for quantum electrodynamics
- Useful calculational tools
- Some examples



7.1 The Dirac Equation

We must include spin



Experimental Tests of QED

★ QED is an incredibly successful theory

Example

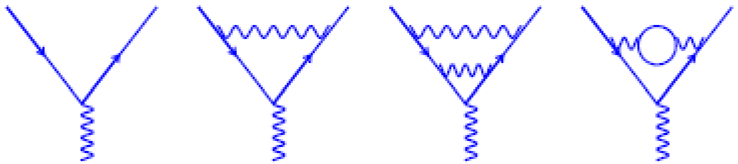
★ Magnetic moments of $e^{\pm}$, $\mu^{\pm}$

$$\vec{\mu} = g\frac{e}{2m}\vec{s}$$

★ For a point-like spin 1/2 particle :

g = 2

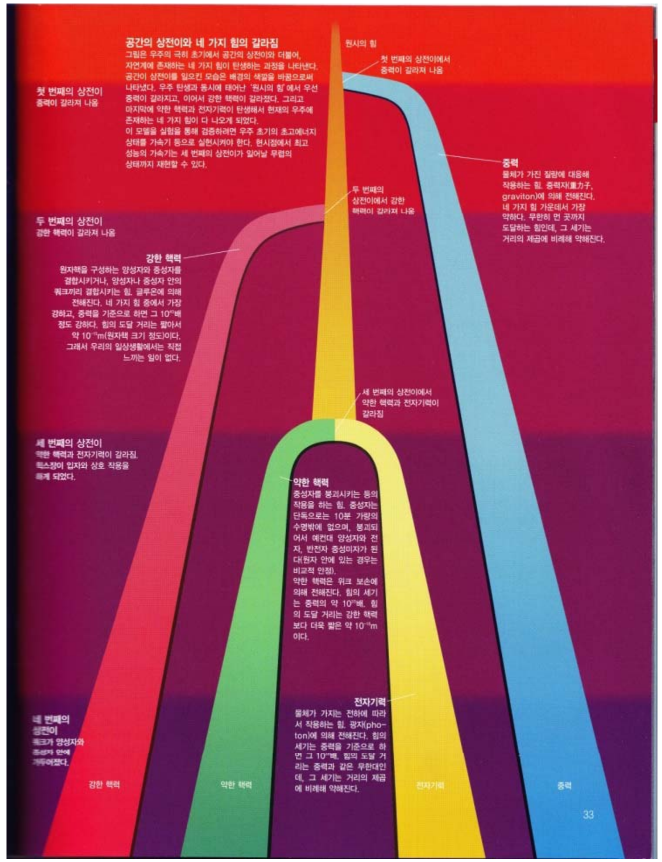
However higher order terms induce an anomalous magnetic moment i.e. g not quite 2.



$$\frac{(g_e\text{-}2)}{2} = 11596521.869 \pm 0.041 \times 10^{-10} \text{ EXPT}$$
$$\frac{(g_e\text{-}2)}{2} = 11596521.3 \pm 0.3 \times 10^{-10} \text{ THEORY}$$

★ Agreement at the level of 1 in 10^8

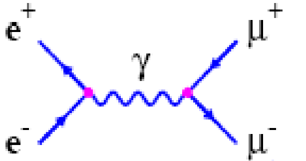
★ Q.E.D. provides a remarkably precise description of the electromagnetic interaction !



Higher Orders

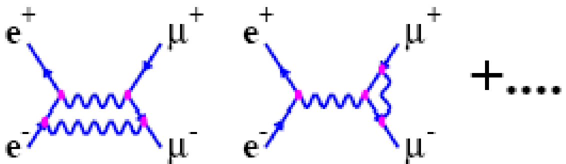
So far only considered lowest order term in the perturbation series. Higher order terms also contribute

Lowest Order:



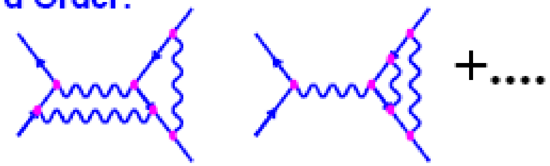
$$|M|^2 \propto \alpha^2 \sim \frac{1}{137^2}$$

Second Order:



$$|M|^2 \propto \alpha^4 \sim \frac{1}{137^4}$$

Third Order:



$$|M|^2 \propto \alpha^6 \sim \frac{1}{137^6}$$

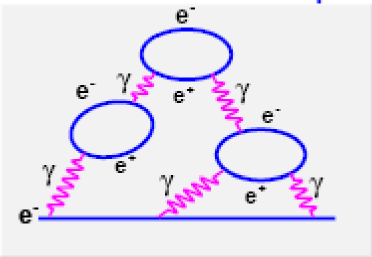
Second order suppressed by α^2 relative to first order. Provided α is small, *i.e.* perturbation is small, lowest order dominates.

Running of α

★ $\alpha = \frac{e^2}{4\pi}$ specifies the strength of the interaction between an electron and photon.

★ BUT α isn't a constant

Consider a free electron: Quantum fluctuations lead to a 'cloud' of virtual electron/positron pairs

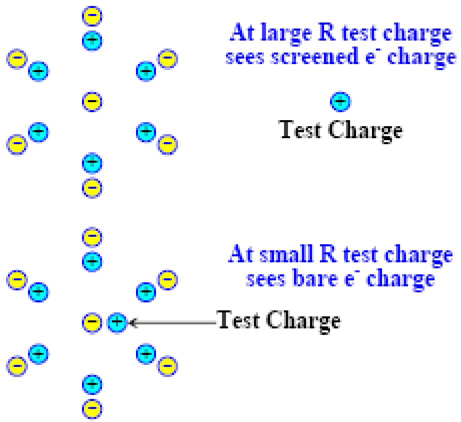


this is just one of many (an infinite set) such diagrams.

★ The vacuum acts like a dielectric medium

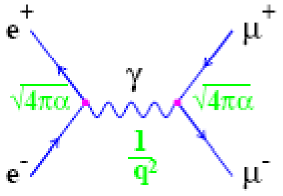
★ The virtual e^+e^- pairs are polarized

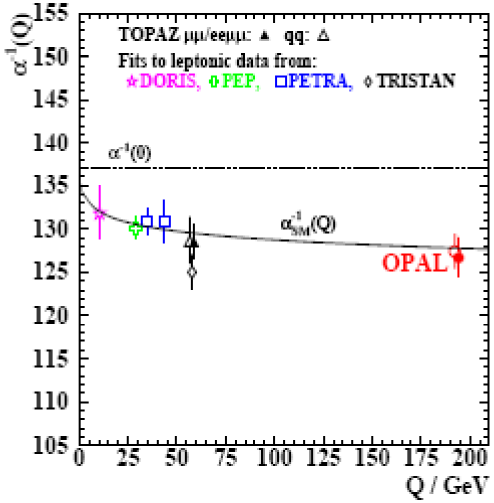
★ At large distances the bare electron charge is screened.



Running of α

Measure $\alpha(q^2)$ from $\mathrm{e^+e^-} \rightarrow \mu^+\mu^-$ etc.





- ★ α increases with the increasing q^2 (i.e. closer to the bare charge).
- ★ At $q^2 = 0$: $\alpha = 1/137$
- ★ At $q^2 = (100\ \mathrm{GeV})^2$: $\alpha = 1/128$

Richard Kass

Weak Interactions

Chapter 8 M&S

Some Weak Interaction basics

Weak force is responsible for β decay e.g. n → pev (1930's)
interaction involves both quarks and leptons
not all quantum numbers are conserved in weak interaction:
- parity, charge conjugation, CP
- isospin
- flavor (strangeness, bottomness, charm)

Weak (+EM) are "completely" described by the Standard Model

Weak interactions has a very rich history

1930's: Fermi's theory described β decay.
1950's: V-A (vector-axial vector) Theory:
- Yang & Lee describe parity violation
- Feynman and Gell-Mann describe muon decay and decay of strange mesons

1960's: Cabibbo Theory

Chapter 8.2.3 M&S

- N. Cabibbo proposes "quark mixing" (1963)
- "explains" why rates for decays with ΔS =0 > ΔS =1

$$BR(K^- \to \mu^- \overline{\nu}_\mu) = 63.5\% \qquad BR(\pi^- \to \mu^- \overline{\nu}_\mu) = 99.9\%$$

Quarks in strong interaction are not the same as the ones in the weak interaction:
weak interaction basis different than strong interaction basis
previous example: $(K^o, \overline{K}^o) \Leftrightarrow (K_s, K_L)$

Weak Interactions

Weinberg-Salam-Glashow (Standard Model 1970's-today)

unify Weak and EM forces

predict neutral current (Z) reactions

gives relationship between mass of W and Z

predict/explain lots of other stuff!..e.g. no flavor changing neutral currents

existence of Higgs ("generates"mass in Standard Model)

Renormalizable Gauge Theory

But the picture is still incomplete:

must input lots of parameters into the Standard Model (e.g. masses)

where's the Higgs and how many are there ?

how many generations of quarks and leptons are there ?

mass pattern of quarks and leptons ?

neutrinos have mass!

CP violation observed with quarks!

is there CP violation with leptons?

7.1 The Dirac Equation

$$\frac{\mathrm{P}^2}{2m} + V = E$$

$$\mathrm{P} \to -i\hbar\nabla, \qquad E \to i\hbar\frac{\partial}{\partial t}$$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$ → Schrödinger's equation

$$E^2 - \mathrm{P}^2c^2 = m^2c^4, \text{ or}$$
$$p^\mu p_\mu - m^2c^2 = 0$$

$$p_\mu \to i\hbar\partial_\mu$$

⬇

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2\psi$$ → Klein-Gorden equation

7.1 The Dirac Equation

- It failed to reproduce the Bohr energy levels for hydrogen because electron has spin ½ but the Klein-Gorden equation applies to particles with spin 0.
- The Klein-Gorden equation is incompatible with Born's statistical interpretation

$$p^0 p_0 - m^2c^2 = \left(p^0 + mc\right)\left(p^0 - mc\right) = 0$$

Spatial part is Zero

$$\left(p^\mu p_\mu - m^2c^2\right) = \left(\beta^\kappa p_\kappa + mc\right)\left(\gamma^\lambda p_\lambda - mc\right)$$

$$\beta^\kappa \gamma^\lambda p_\kappa p_\lambda - mc\left(\beta^\kappa - \gamma^\kappa\right)p_\kappa - m^2c^2$$

We don't want this term

$$\Rightarrow \quad \beta^\kappa = \gamma^\kappa$$

7.1 The Dirac Equation

$$p^\mu p_\mu = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$$

↓

$$\begin{aligned}\left(p^0\right)^2 - \left(p^1\right)^2 - \left(p^2\right)^2 - \left(p^3\right)^2 = {} & \left(\gamma^0\right)^2\left(p^0\right)^2 + \left(\gamma^1\right)^2\left(p^1\right)^2 + \left(\gamma^2\right)^2\left(p^2\right)^2 \\ & + \left(\gamma^3\right)^2\left(p^3\right)^2 + \left(\gamma^0\gamma^1 + \gamma^1\gamma^0\right)p^0 p^1 \\ & + \left(\gamma^0\gamma^2 + \gamma^2\gamma^0\right)p^0 p^2 + \left(\gamma^0\gamma^3 + \gamma^3\gamma^0\right)p^0 p^3 \\ & + \left(\gamma^1\gamma^2 + \gamma^2\gamma^1\right)p^1 p^2 + \left(\gamma^1\gamma^3 + \gamma^3\gamma^1\right)p^1 p^3 \\ & + \left(\gamma^2\gamma^3 + \gamma^3\gamma^2\right)p^2 p^3\end{aligned}$$

↓

$$\left\{\gamma^\mu, \gamma^\nu\right\} = 2g^{\mu\nu}$$

7.1 The Dirac Equation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Pauli matrix

$$\left(p^\mu p_\mu - m^2c^2\right) = \left(\gamma^\kappa p_\kappa + mc\right)\left(\gamma^\lambda p_\lambda - mc\right) = 0$$

Pick one

$$\gamma^\lambda p_\lambda - mc = 0$$

$$i\hbar\gamma^\mu \partial_\mu \psi - mc\psi = 0$$

Dirac Equation

Dirac spinor

7.2 Solutions to the Dirac Equation

$$\text{suppose} \quad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial z} = 0$$

Independent of position

$$\frac{i\hbar}{c}\gamma^0 \frac{\partial \psi}{\partial t} - mc\psi = 0$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial \psi_A / \partial t \\ \partial \psi_B / \partial t \end{pmatrix} = -i \frac{mc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

where

$$\psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{upper component}$$

$$\psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \quad \text{lower component}$$

7.2 Solutions to the Dirac Equation

$$\frac{\partial \psi_A}{\partial t} = -i\left(\frac{mc^2}{\hbar}\right)\psi_A, \qquad -\frac{\partial \psi_B}{\partial t} = -i\left(\frac{mc^2}{\hbar}\right)\psi_B$$

$$\psi_A(t) = e^{-i\left(mc^2/\hbar\right)t}\psi_A(0), \qquad \psi_B(t) = e^{+i\left(mc^2/\hbar\right)t}\psi_B(0)$$

State of particle

State of Antiparticle

$e^{-iEt/\hbar}$

$$\psi^{(1)} = e^{-i\left(mc^2/\hbar\right)t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \psi^{(2)} = e^{-i\left(mc^2/\hbar\right)t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\psi^{(3)} = e^{+i\left(mc^2/\hbar\right)t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \psi^{(4)} = e^{+i\left(mc^2/\hbar\right)t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

7.2 Solutions to the Dirac Equation

We look next for plane-wave solutions

$$\psi(x) = ae^{-ik\cdot x}u(k)$$

Bispinor
Four-vector

$$\partial_\mu \psi = -ik_\mu \psi$$

$$\hbar\gamma^\mu k_\mu e^{-ik\cdot x}u - mce^{-ik\cdot x}u = 0$$

or

$$\left(\hbar\gamma^\mu k_\mu - mc\right)u = 0$$

7.2 Solutions to the Dirac Equation

$$\gamma^{\mu} k_{\mu} = \gamma^{0} k_{0} - \gamma \cdot k = k^{0} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - k \cdot \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} = \begin{pmatrix} k^{0} & -k \cdot \sigma \\ k \cdot \sigma & -k^{0} \end{pmatrix}$$

or

$$\left(\hbar \gamma^{\mu} k_{\mu} - mc\right) u = \begin{pmatrix} (\hbar k^{0} - mc) & -\hbar k \cdot \sigma \\ \hbar k \cdot \sigma & (-\hbar k^{0} - mc) \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$= \begin{pmatrix} (\hbar k^{0} - mc) u_A - \hbar k \cdot \sigma u_B \\ \hbar k \cdot \sigma u_A - (\hbar k^{0} + mc) u_B \end{pmatrix}$$

7.2 Solutions to the Dirac Equation

$$u_A = \frac{1}{k^0 - mc/\hbar}(k \cdot \sigma)u_B \quad \text{and} \quad u_B = \frac{1}{k^0 + mc/\hbar}(k \cdot \sigma)u_A$$

$$u_A = \frac{1}{(k^0)^2 - (mc/\hbar)^2}(k \cdot \sigma)^2 u_A$$

$$(k \cdot \sigma)^2 = \begin{pmatrix} k_z^2 + (k_x - ik_y)(k_x + ik_y) & k_z(k_x - ik_y) - k_z(k_x - ik_y) \\ k_z(k_x - ik_y) - k_z(k_x - ik_y) & (k_x - ik_y)(k_x + ik_y) + k_z^2 \end{pmatrix} = k^2 1$$

7.2 Solutions to the Dirac Equation

$$u_A = \frac{k^2}{(k^0)^2 - (mc/\hbar)^2} u_A$$

and

$$(k^0)^2 - (mc/\hbar)^2 = k^2 \quad \text{or} \quad k^2 = k^\mu k_\mu = (mc/\hbar)^2$$

Relate to the energy-momentum four-vector

$$k^\mu = \pm p^\mu / \hbar$$

Positive sign -> particle state
Negative sign -> antiparticle state

7.2 Solutions to the Dirac Equation

$$N = \sqrt{(E+mc^2)/c}$$

$$u^{(1)} = N\begin{pmatrix} 1 \\ 0 \\ \dfrac{c(p_z)}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)} = N\begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_z)}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

and

$$u^{(3)} = N\begin{pmatrix} \dfrac{c(p_x-ip_y)}{E+mc^2} \\ \dfrac{c(-p_z)}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}, \quad u^{(4)} = -N\begin{pmatrix} \dfrac{c(p_z)}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$\psi = ae^{-ip\cdot x/\hbar}u \quad (particles), \quad \psi = ae^{ip\cdot x/\hbar}v \quad (antiparticles)$$

7.3 Bilinear Covariants

The components of a Dirac spinor do not transform as a four-vector

If you go to a system moving with speed v in the x direction

$$\psi \to \psi' = S\psi$$

$$S = a_+ + a_- \gamma^0 \gamma^1 = \begin{pmatrix} a_+ & a_- \sigma_1 \\ a_- \sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{pmatrix}$$

$$a_\pm = \pm\sqrt{\tfrac{1}{2}(\gamma \pm 1)}$$

7.3 Bilinear Covariants

We want to construct a scalar quantity out of a spinor

$$\psi^{\dagger}\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

$$\left(\psi^{\dagger}\psi\right)' = (\psi')^{\dagger}\psi' = \psi^{\dagger}S^{\dagger}S\psi \neq \left(\psi^{\dagger}\psi\right)$$

$$S^{\dagger}S = \gamma\begin{pmatrix} 1 & -(v/c)\sigma_1 \\ -(v/c)\sigma_1 & 1 \end{pmatrix} \neq 1$$

7.3 Bilinear Covariants

Introduce the adjoint spinor

$$\overline{\psi} \equiv \psi^{\dagger}\gamma^{0} = \psi_1^{*} + \psi_2^{*} - \psi_3^{*} - \psi_4^{*}$$

$$\left(\overline{\psi}\psi\right) = \psi^{\dagger}\gamma^{0}\psi = \left|\psi_1\right|^{2} + \left|\psi_2\right|^{2} - \left|\psi_3\right|^{2} - \left|\psi_4\right|^{2}$$

$$\left(\overline{\psi}\psi\right)' = \left(\psi'\right)^{\dagger}\gamma^{0}\psi' = \psi^{\dagger}S^{\dagger}\gamma^{0}S\psi = \psi^{\dagger}\gamma^{0}\psi = \overline{\psi}\psi$$

7.3 Bilinear Covariants

Parity transformation

$$P:\ (x, y, z) \rightarrow (-x, -y, -z)$$

$$\psi \rightarrow \psi' = \gamma^0 \psi$$

$$\left(\overline{\psi}\psi\right)' = (\psi')^\dagger \gamma^0 \psi' = \psi^\dagger \left(\gamma^0\right)^\dagger \gamma^0 \gamma^0 \psi = \psi^\dagger \gamma^0 \psi = \overline{\psi}\psi$$

Invariant under transformation

True scalar

We can make a pseudoscalar

$$\overline{\psi}\gamma^5\psi$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

7.4 The Photon

Maxwell's equations

$$\begin{cases} \text{(i)} \quad \nabla \cdot E = 4\pi\rho & \text{(iii)} \quad \nabla \cdot B = 0 \\ \text{(ii)} \quad \nabla \times E + \dfrac{1}{c}\dfrac{\partial B}{\partial t} = 0 & \text{(iv)} \quad \nabla \times B - \dfrac{1}{c}\dfrac{\partial E}{\partial t} = \dfrac{4\pi}{c} J \end{cases}$$

Field strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$J^{\mu} = \left(c\rho, J\right)$$

7.4 The Photon

$$\partial_{\mu} F^{\mu\nu} = \frac{4\pi}{c} J^{\nu}$$

$$\partial_{\mu} J^{\mu} = 0$$

antisymmetry

$$B = \nabla \times A$$

$$\nabla \times \left(E + \frac{1}{c}\frac{\partial A}{\partial t} \right) = 0$$

$$E = -\nabla V - \frac{1}{c}\frac{\partial A}{\partial t}$$

$$A^{\mu} = (V, A)$$

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$

7.4 The Photon

$$\partial^{\mu}\partial_{\mu}A^{\nu}-\partial^{\nu}\left(\partial_{\mu}A^{\mu}\right)=\frac{4\pi}{c}J^{\nu}$$

$$A'_{\mu}=A_{\mu}+\partial_{\mu}\lambda$$

Any function of position and time

$$\partial^{\mu}A^{\nu'}-\partial^{\nu}A^{\mu'}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}$$

Gauge transformation

$$\partial_{\mu}A^{\mu}=0$$

Lorentz condition

7.4 The Photon

$$\Box A^{\mu} = \frac{4\pi}{c} J^{\mu}$$

$$\Box \equiv \partial^{\mu}\partial_{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

In QED, vector potential becomes the wave function of the photon

$$J^{\mu} = 0$$

$$\Box A^{\mu} = 0$$

Free photon

Plane-wave

$$A^{\mu}(x) = a e^{-(i/\hbar)p\cdot x} \varepsilon^{\mu}(p)$$

Polarization

7.4 The Photon

$$p^{\mu}\varepsilon_{\mu} = 0$$

Lorentz condition

$$\Downarrow$$

Coulomb gauge

$$\varepsilon^{0} = 0, \quad \text{so} \;\; \varepsilon \cdot p = 0$$

7.5 The Feynman Rules for QED

Electrons

$$\psi(x) = ae^{-(i/\hbar)p\cdot x}u^{(s)}(p)$$

$$(\gamma^{\mu}p_{\mu} - mc)u = 0$$

$$\bar{u}(\gamma^{\mu}p_{\mu} - mc) = 0$$

$$\overline{u^{(1)}}u^{(2)} = 0$$

$$\bar{u}u = 2mc$$

$$\sum_{s=1,2}\bar{u}^{(s)}u^{(s)} = (\gamma^{\mu}p_{\mu} + mc)$$

Positrons

$$\psi(x) = ae^{(i/\hbar)p\cdot x}v^{(s)}(p)$$

$$(\gamma^{\mu}p_{\mu} - mc)v = 0$$

$$\bar{v}(\gamma^{\mu}p_{\mu} - mc) = 0$$

$$\overline{v^{(1)}}v^{(2)} = 0$$

$$\bar{v}v = 2mc$$

$$\sum_{s=1,2}\bar{v}^{(s)}v^{(s)} = (\gamma^{\mu}p_{\mu} - mc)$$

7.5 The Feynman Rules for QED

Photons

$$A_\mu(x) = a e^{-(i/\hbar) p \cdot x} \varepsilon_\mu^{(s)}$$

$$p^\mu \varepsilon_\mu = 0$$

$$\varepsilon_\mu^{(1)^*} \varepsilon^{(2)\mu} = 0$$

$$\varepsilon^{\mu^*} \varepsilon_\mu = -1$$

$$\varepsilon^0 = 0, \quad \text{so} \ \ \varepsilon \cdot p = 0$$

$$\sum_{s=1,2} \varepsilon_i^{(s)} \varepsilon_j^{(s)*} = \delta_{ij} - \hat{p}_i \hat{p}_j$$

7.5 The Feynman Rules for QED

Feynman Rules

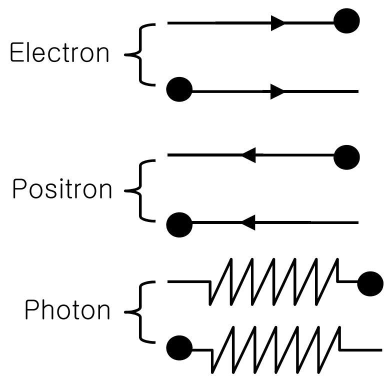
1. Notation

Associate a momentum.

Draw an arrow next to the line

Indicating the positive direction (forward in time)

2. External lines.



7.5 The Feynman Rules for QED

Feynman Rules

3. Vertex Factors

$$ig_e\gamma^\mu \qquad g_e = e\sqrt{4\pi/\hbar c} = \sqrt{4\pi\alpha}$$

4. Propagators

Electrons and Positrons

$$\frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$$

Photons

$$\frac{-ig_{\mu\nu}}{q^2}$$

7.5 The Feynman Rules for QED

Feynman Rules

5. Conservation of energy and momentum

$$(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$$

6. Integrate over internal momenta

$$\frac{d^4 q}{(2\pi)^4}$$

7. Cancel the delta function

7.5 The Feynman Rules for QED

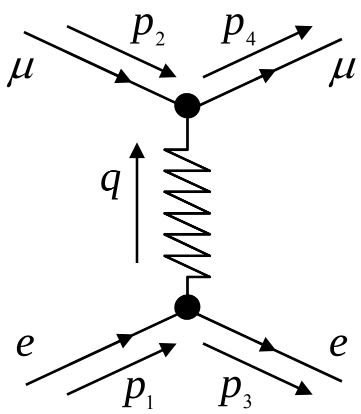
Feynman Rules

8. Antisymmetrization

Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) electrons (or positrons), or of an incoming electron with and outgoing positron (or vice versa).

7.6 Examples

Electron-Muon Scattering

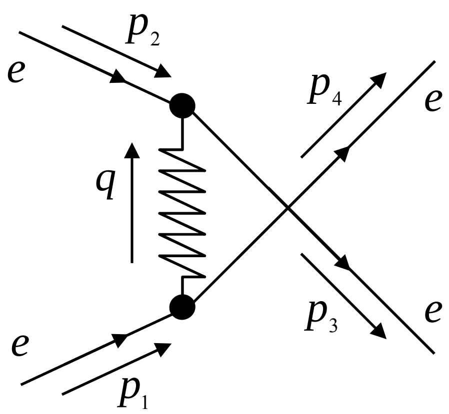


$$(2\pi)^4 \int [\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)u^{(s_1)}(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}^{(s_4)}(p_4)(ig_e\gamma^\nu)u^{(s_2)}(p_2)] \times \delta^4(p_1 - p_3 - q)\delta^4(p_2 + q - p_4)d^4q$$

$$M = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)]$$

7.6 Examples

Electron-Electron Scattering

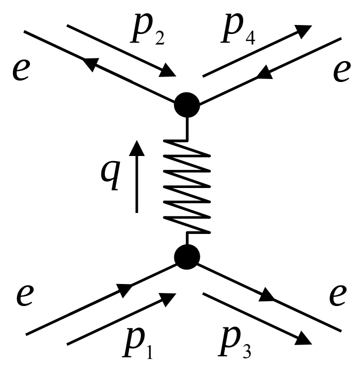


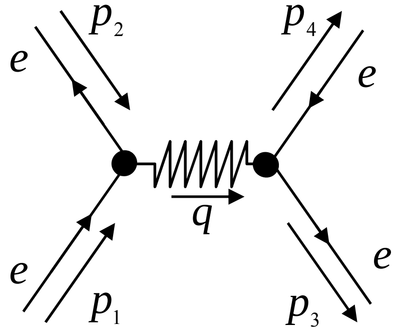
$$M = -\frac{g_e^{\,2}}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^{\mu}u(1)][\bar{u}(4)\gamma_{\mu}u(2)] + \frac{g_e^{\,2}}{(p_1 - p_4)^2}[\bar{u}(4)\gamma^{\mu}u(1)][\bar{u}(3)\gamma_{\mu}u(2)]$$

Rule 8

7.6 Examples

Electron-Positron Scattering





7.6 Examples

Electron-Positron Scattering

$$M_1 = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{v}(2)\gamma_\mu v(4)]$$

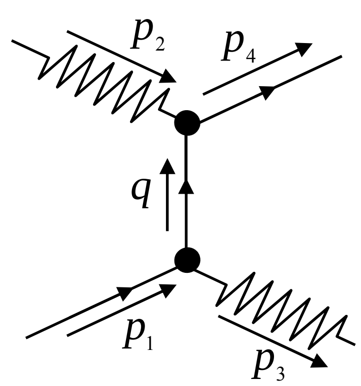
$$M_2 = -\frac{g_e^2}{(p_1 + p_2)^2}[\bar{u}(3)\gamma^\mu v(4)][\bar{v}(2)\gamma_\mu u(1)]$$

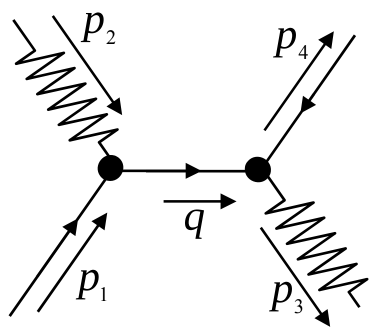
$$M = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{v}(2)\gamma_\mu v(4)] + \frac{g_e^2}{(p_1 + p_2)^2}[\bar{u}(3)\gamma^\mu v(4)][\bar{v}(2)\gamma_\mu u(1)]$$

Rule 8

7.6 Examples

Compton scattering





7.6 Examples

Compton scattering

$$(2\pi)^4 \int \varepsilon_\mu(2)[\bar{u}(4)(ig_e\gamma^\mu)\frac{i(\not{q}+mc)}{(q^2-m^2c^2)}(ig_e\gamma^\nu)u(1)]\varepsilon_\nu(3)^* \times \delta^4(p_1-p_3-q)\delta^4(p_2+q-p_4)d^4q$$

$$\not{a} = a^\mu\gamma_\mu$$

$$M_1 = \frac{g_e^2}{(p_1-p_3)^2-m^2c^2}[\bar{u}(4)\not{\varepsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\varepsilon}(3)^*u(1)]$$

$$M_2 = \frac{g_e^2}{(p_1+p_2)^2-m^2c^2}[\bar{u}(4)\not{\varepsilon}(3)^*(\not{p}_1+\not{p}_2+mc)\not{\varepsilon}(2)u(1)]$$

$$M = M_1 + M_2$$

7.7 Casimir's Trick

$$\langle |M|^2 \rangle \equiv \textit{average} \text{ initial spins, } \textit{sum} \text{ over final spins,}$$
$$\text{of } |M(s_i \to s_f)|^2$$

Electron-Muon

$$|M|^2 = \frac{g_e^{\ 4}}{(p_1 - p_3)^4} [\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)][\bar{u}(3)\gamma^\nu u(1)]^*[\bar{u}(4)\gamma_\nu u(2)]^*$$

Try this

$$G \equiv [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^*$$

$$[\bar{u}(a)\Gamma_2 u(b)]^* = [u(a)^\dagger \gamma^0 \Gamma_2 u(b)]^\dagger = u(b)^\dagger \Gamma_2^{\ \dagger} \gamma^{0\dagger} u(a)$$

$$\gamma^{0\dagger} = \gamma^0,\ (\gamma^0)^2 = 1$$

$$u(b)^\dagger \Gamma_2^{\ \dagger} \gamma^{0\dagger} u(a) = u(b)^\dagger \gamma^0 \gamma^0 \Gamma_2^{\ \dagger} \gamma^0 u(a) = \bar{u}(b)\bar{\Gamma}_2 u(a)$$

7.7 Casimir's Trick

$$G \equiv [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(b)\bar{\Gamma}_2 u(a)]$$

$$\begin{aligned}\sum_{b\text{ spins}} G &= \bar{u}(a)\Gamma_1 \left\{ \sum_{s_b=1,2} u^{(s_b)}(p_b)\bar{u}^{(s_b)}(p_b) \right\} \bar{\Gamma}_2 u(a) \\ &= \bar{u}(a)\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2 u(a) = \bar{u}(a) Q u(a)\end{aligned}$$

$$\begin{aligned}\sum_{a\text{ spins}} \sum_{b\text{ spins}} G &= \sum_{s_a=1,2} \bar{u}^{(s_a)}(p_a) Q u^{(s_a)}(p_a) \\ &= \sum_{s_a=1,2} \sum_{i,j=1}^{4} \bar{u}^{(s_a)}(p_a)_i Q_{ij} u^{(s_a)}(p_a)_j = \sum_{i,j=1}^{4} Q_{ij} \left\{ \sum_{s_a=1,2} u^{(s_a)}(p_a)\bar{u}^{(s_a)}(p_a) \right\}_{ji} \\ &= \sum_{i,j=1}^{4} Q_{ij}(\not{p}_a + m_a c)_{ji} = \sum_{i=1}^{4} [Q(\not{p}_a + m_a c)]_{ii} = Tr[Q(\not{p}_a + m_a c)]\end{aligned}$$

7.7 Casimir's Trick

Conclusion

$$\sum_{\text{all spins}} [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^* = Tr[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)]$$

There are no spinors left!

Average

$$\langle |M|^2 \rangle = \frac{g_e^4}{4(p_1 - p_3)^4} Tr[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)] \times Tr[\gamma_\mu(\not{p}_2 + Mc)\gamma_\nu(\not{p}_4 + Mc)]$$

Electron

Muon

7.8 Cross Sections and Lifetimes

Mott and Rutherford scattering

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi Mc}\right)^2 \left\langle |M|^2 \right\rangle$$

$$p_1 = (E/c, p_1) \qquad p_2 = (Mc, 0) \qquad p_3 = (E/c, p_3) \qquad p_4 = (Mc, 0)$$

$$(p_1 - p_3)^2 = -(p_1 - p_3)^2 = -4p^2 \sin^2(\theta/2)$$

$$p_1 \cdot p_3 = m^2c^2 + 2p^2 \sin^2(\theta/2)$$

$$(p_1 \cdot p_2)(p_3 \cdot p_4) = (p_1 \cdot p_4)(p_2 \cdot p_3) = (ME)^2$$

$$(p_2 \cdot p_4) = (Mc)^2$$

7.8 Cross Sections and Lifetimes

Mott and Rutherford scattering

$$\left\langle |M|^2 \right\rangle = \left(\frac{g_e^2 Mc}{p^2 \sin^2(\theta/2)} \right)^2 [(mc)^2 + p^2 \cos^2(\theta/2)]$$

⬇

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar}{2p^2 \sin^2(\theta/2)} \right)^2 [(mc)^2 + p^2 \cos^2(\theta/2)]$$