Practical Computation of Physically Measurable Quantities

Syllabus

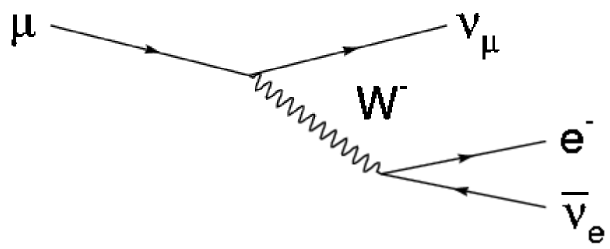
- Introduction (Chap. 1)
- Special Relativity (Chap. 2)
- Quantum Mechanics (Chap. 3)
  - Hydrogen Atom
  - Meson wave function
  - Baryon wave function
  - Magnetic moments
- Detector
- Data Processing
- Feynman diagram (Chap. 4)
- QED (Chap. 5)
- QCD (Chap. 6)
- Weak interaction (Chap. 7)

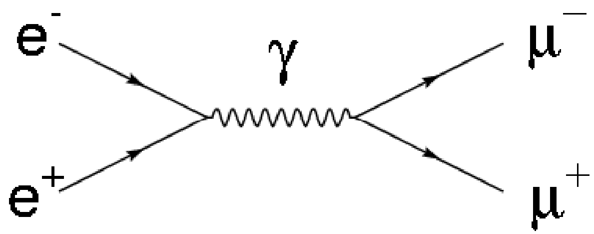
In this chapter.....

- How to study decay and scattering problems by introducing Feynman diagram techniques

Feynman Diagrams

Feynman diagrams are pictorial representations of AMPLTUDES of particle reactions, i.e scatterings or decays. Use of Feynman diagrams can greatly reduce the amount of computation involved in calculating a rate or
cross section of a physical process, e.g.
muon decay: $\mu^- \to e^- \nu_e \nu_\mu$ or $e^+e^- \to \mu^+\mu^-$ scattering.





Feynman and his diagrams

Like electrical circuit diagrams, every line in the diagram has a strict mathematical interpretation. Unfortunately the mathematical overhead necessary to do complete calculations with this technique is large and there is not enough time in this course to go through all the details. The details of Feynman diagrams are addressed in Advanced course. For a taste and summary of the rules look at Griffiths (e.g. sections 6.3, 6.6, and 7.5) or Relativistic Quantum Mechanics by Bjorken & Drell.

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Feynman Diagrams

Each Feynman diagram represents an AMPLITUDE (M).

Quantities such as cross sections and decay rates (lifetimes) are proportional to $|M|^2$.

The transition rate for a process can be calculated using time dependent perturbation theory using Fermi's Golden Rule:

$$transition\ rate = \frac{2\pi}{\hbar}|M|^2 \times (phase\ space)$$

In lowest order perturbation theory M is the fourier transform of the potential. M&S B.20-22, p295
"Born Approximation" M&S 1.27, p17

The differential cross section for two body scattering (e.g. $pp \rightarrow pp$) in the CM frame is:

M&S B.29, p296

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}\frac{q_f^2}{v_i v_f}|M|^2$$

q_f=final state momentum
v_f= speed of final state particle
v_i= speed of initial state particle

The decay rate (Γ) for a two body decay (e.g. $K^0 \rightarrow \pi^+\pi^-$) in CM is given by:

Griffiths 6.32

$$\Gamma = \frac{S|\vec{p}|}{8\pi\hbar m^2 c}|M|^2$$

m=mass of parent
p=momentum of decay particle
S=statistical factor (fermions/bosons)

In most cases $|M|^2$ cannot be calculated exactly.
Often M is expanded in a power series.
Feynman diagrams represent terms in the series expansion of M.

Feynman Diagrams

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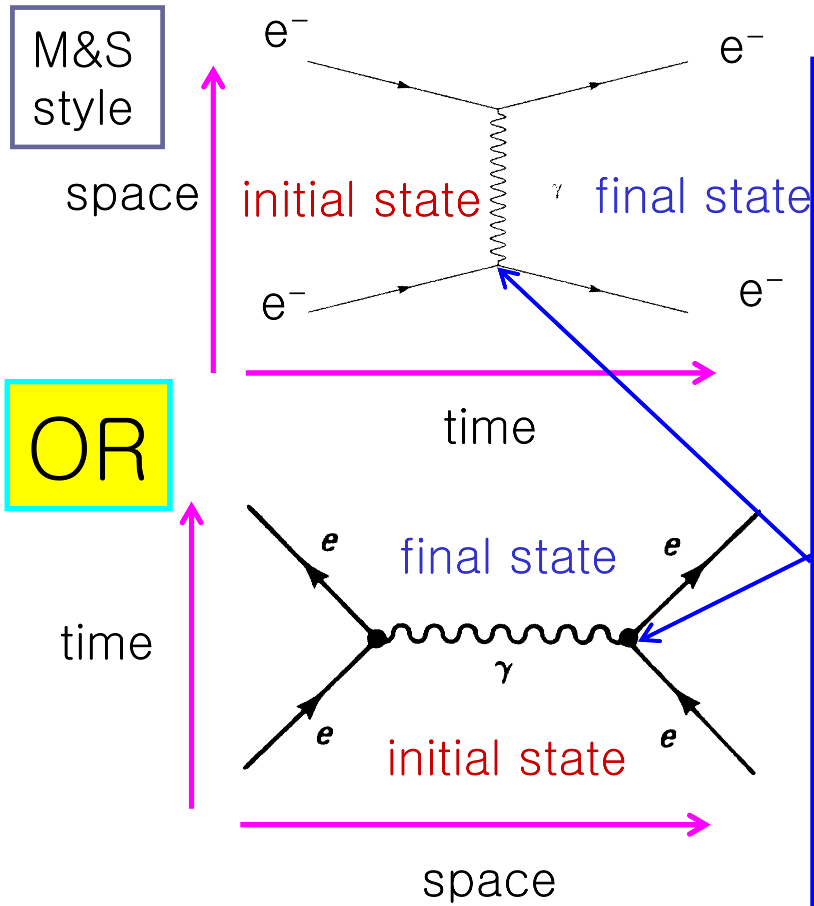
Feynman diagrams plot time vs space: Moller Scattering $e^-e^- \to e^-e^-$



QED Rules

Solid lines are charged fermions

electrons or positrons (spinor wavefunctions)

Wavy (or dashed) lines are photons

Arrow on solid line signifies e^- or e^+

e^- arrow in same direction as time

e^+ arrow opposite direction as time

At each vertex there is a coupling constant

$\sqrt{\alpha}$, α= 1/137=fine structure constant

Quantum numbers are conserved at a vertex

e.g. electric charge, lepton number

"Virtual" Particles do not conserve E, **p**

virtual particles are internal to diagram(s)

for γ's: $E^2-\mathbf{p}^2\neq 0$ (off "mass shell")

in all calculations we integrate over the virtual particles 4-momentum (4d integral)

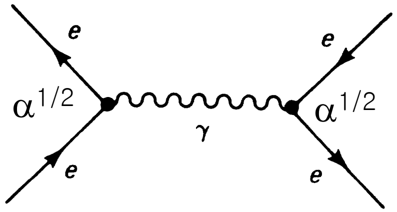
Photons couple to electric charge

no photons only vertices

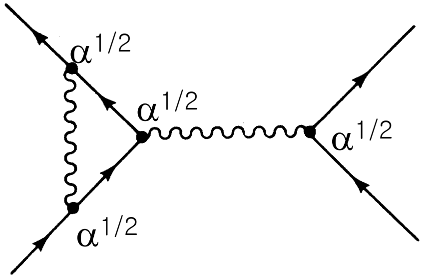
Feynman Diagrams

We classify diagrams by the *order* of the coupling constant:

Bhabha scattering: $e^+e^- \rightarrow e^+e^-$



Amplitude is of order α.



Amplitude is of order α^2.

Since α_{QED} =1/137 higher order diagrams should be corrections to lower order diagrams.

This is just perturbation Theory!!

- This expansion in the coupling constant works for QED since α_{QED}=1/137
- Does not work well for QCD where $\alpha_{QCD} \approx 1$

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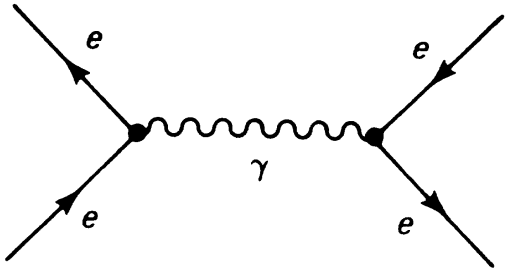
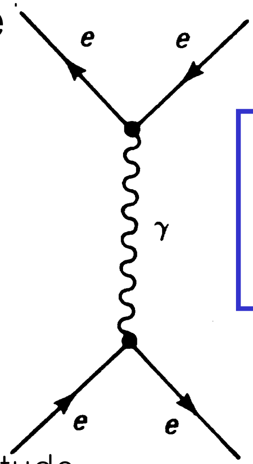
Feynman Diagrams

For a given order of the coupling constant there can be many diagrams

Bhabha scattering: $e^+e^-\rightarrow e$



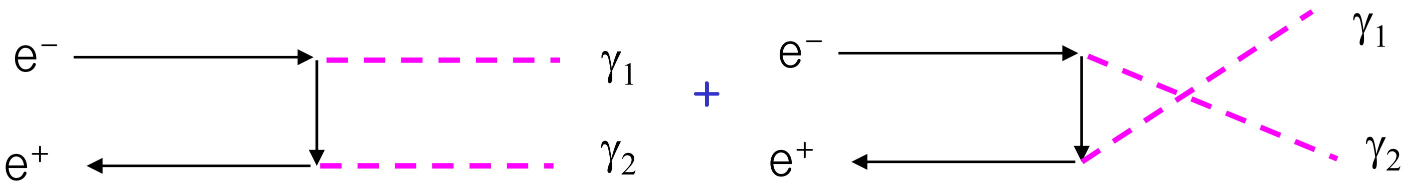


amplitudes can interfere constructively or destructively

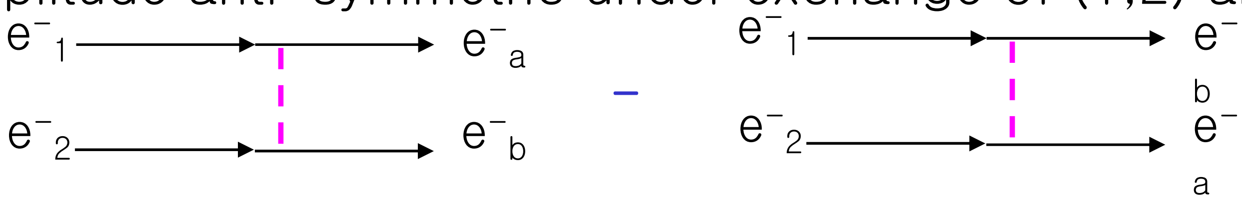
Must add/subtract diagram together to get the total amplitude

total amplitude must reflect the symmetry of the process

$e^+e^-\rightarrow\gamma\gamma$ identical bosons in final state, amplitude symmetric under exchange of γ_1, γ_2.



Moller scattering: $e^-e^-\rightarrow e^-e^-$ identical fermions in initial and final state amplitude anti-symmetric under exchange of (1,2) and (a,b)



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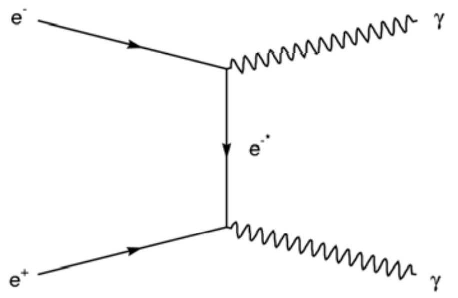
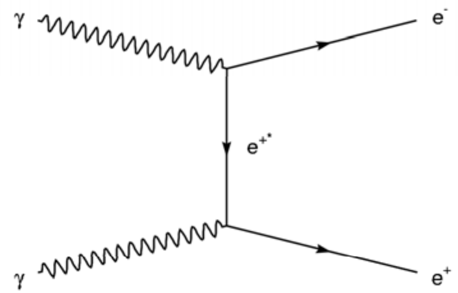
Feynman Diagrams

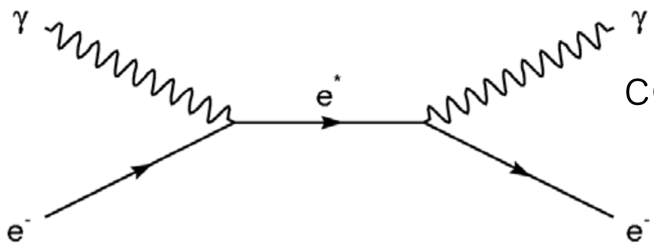
Feynman diagrams of a given order are related to each other!



$e^+e^- \rightarrow \gamma\gamma$ γ's in final state



$\gamma\gamma \rightarrow e^+e^-$ γ's in inital state



compton scattering
$\gamma\, e^- \rightarrow \gamma\, e^-$

electron and positron wavefunctions are related to each other.

Rates and Cross Sections

For reaction

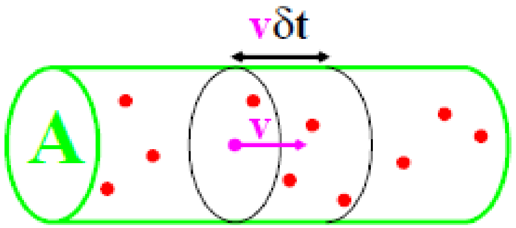
$$\mathrm{a} + \mathrm{b} \rightarrow c + d$$

The cross section, σ, is defined as the reaction rate per target particle, Γ, per unit incident flux, ϕ

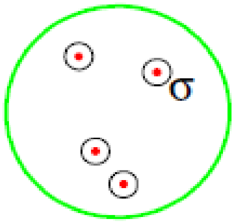
$$\Gamma = \phi\sigma$$

where Γ is given by Fermi's Golden Rule.

Example: Consider a single particle of type a traversing a beam (area A) of particles of type b of number density n_b.



In time δt traverses a region containing $v\delta t A n_b$ particles of type b.



Interaction probability defined as effective cross sectional area occupied by the $v\delta t A n_b$ particles of type b

$$\frac{v\delta t A n_b \sigma}{A} = v\delta t n_b \sigma$$

Therefore the reaction rate is $v n_b \sigma$.

(see Question 2 on the problem sheet)

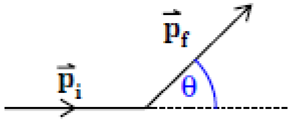
Scattering in Q.M.

REVISION (see Dr Ritchie's QMII transparencies 11.8-11.15)

NOTE: Natural Units used throughout $\hbar = c = 1$

$$\tilde{p} = \hbar\,\tilde{k} \rightarrow \tilde{p} = \tilde{k} \quad etc.$$

Consider a beam of particles scattering in Potential $V(r)$



★ Scattering rate characterized by the interaction cross section σ

$$\sigma = \frac{\text{number of particles scattered/unit time}}{\text{incident flux}}$$

Use FERMI'S GOLDEN RULE for Transition rate, Γ:

$$\Gamma = 2\pi|M|^2\rho(E_f)$$

where M is the Matrix Element and $\rho(E_f)$ = density of final states.

★ 1st Order Perturbation Theory using plane wave solutions of form $\psi = N e^{-i(Et - \tilde{p}.\tilde{r})}$.

Require :

- wave-function normalization
- matrix element in perturbation theory
- expression for flux
- expression for density of states

Normalization: Normalize wave-functions to one particle in a box of side L

$$|\psi_i|^2 = N^2 = 1/L^3$$
$$N = (1/L)^{\frac{3}{2}}$$

Matrix Element: this contains the physics of the interaction

$$M = \langle\psi_f|\hat{\mathrm{H}}|\psi_i\rangle$$
$$M = \int \psi_f^* \hat{\mathrm{H}}\psi_i d^3\tilde{\mathrm{r}}$$
$$M = \int N e^{-i\tilde{\mathrm{p}}_{\mathrm{f}}.\tilde{\mathrm{r}}} V(\tilde{\mathrm{r}}) N e^{i\tilde{\mathrm{p}}_{\mathrm{i}}.\tilde{\mathrm{r}}} d^3\tilde{\mathrm{r}}$$
$$M = \frac{1}{L^3}\int e^{i\tilde{\mathrm{p}}.\tilde{\mathrm{r}}} V(\tilde{\mathrm{r}}) d^3\tilde{\mathrm{r}}$$

where $\tilde{\mathrm{p}} = \tilde{\mathrm{p}}_{\mathrm{i}} - \tilde{\mathrm{p}}_{\mathrm{f}}$

Incident Flux: Consider a “target” of area A and a beam of particles traveling at $v = c$ towards the target. Any incident particle with in a volume cA will cross the target area every second. Flux = number of incident particles crossing unit area per second :

$$\text{flux} = \frac{cA}{A} n_i = c n_i$$

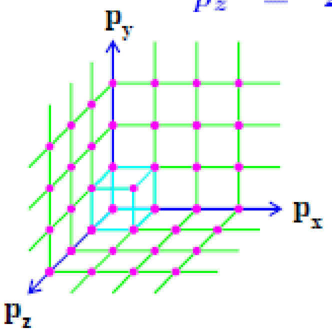
where n_i is number density of incident particles = 1 per L^3

$$\text{flux} = c/L^3 = 1/L^3 \quad (c = 1)$$

$$\text{Flux} = \frac{\text{number of incident particle}}{A\ t}$$

Density of states: for box of side L states are given by periodic boundary conditions:

$$\begin{aligned} k_x &= 2\pi n_x/L, \;\; etc. \\ \Rightarrow \quad p_x &= 2\pi n_x/L \quad (\hbar = 1) \\ p_y &= 2\pi n_y/L \\ p_z &= 2\pi n_z/L \end{aligned}$$



Volume of single state in momentum space:

$$\left(\frac{2\pi}{L}\right)^3$$

Number of final states between $p \to p + dp$:

$$\begin{aligned} dN &= p^2 dp d\Omega/(2\pi/L)^3 \\ \therefore \rho(p_f) &= dN/dp = p^2 d\Omega/(2\pi/L)^3 \end{aligned}$$

In almost all scattering process considered in these lectures the final state particles have $E \gg m$ and to a good approximation $E^2 = p^2 + m^2 \;\; \rightarrow E = p$.

$$\begin{aligned} \rho(E) &= \frac{dN}{dE} = \frac{dN}{dp}\frac{dp}{dE} \\ &= E^2 d\Omega/(2\pi/L)^3 \end{aligned}$$

$$\rho(E) = \frac{E^2 d\Omega}{(2\pi)^3} L^3$$

★ Putting all the separate bits together:

$$d\sigma = \frac{1}{\text{flux}} 2\pi |M|^2 \rho(E_f)$$

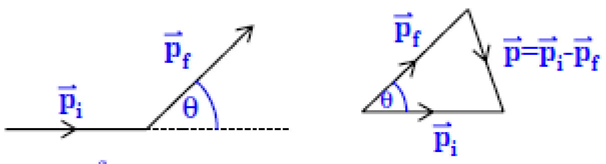
$$= L^3 2\pi \left| \frac{1}{L^3} \int e^{i\tilde{\mathrm{p}}.\tilde{\mathrm{r}}} V(r) d^3\tilde{\mathrm{r}} \right|^2 E^2 \left(\frac{L}{2\pi}\right)^3 d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2} \left| \int e^{i\tilde{\mathrm{p}}.\tilde{\mathrm{r}}} V(r) d^3\tilde{\mathrm{r}} \right|^2$$

The normalization cancels and, in the limit where the incident particles have $v \approx c$ and the out-going particles have $E \gg m \to E_f = p_f$, arrive at a simple expression. Apply to the elastic scattering of a particle from in a Yukawa potential.

Scattering from Yukawa Potential

$$V(r) = -\frac{g^2}{4\pi} \frac{e^{-mr}}{r}$$



$$M_{fi} = \int e^{i\tilde{\mathrm{p}}.\tilde{\mathrm{r}}} V(r) d^3\tilde{\mathrm{r}}$$

$$M_{fi} = -\frac{g^2}{4\pi} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{i|\tilde{\mathrm{p}}|r\cos\theta'} \frac{e^{-mr}}{r} r^2 \sin\theta' dr d\theta' d\phi$$

Where for the purposes of the integration, the z-axis is been defined to lie in the direction of $\tilde{\mathrm{p}}$ and θ' is the polar angle with respect to this axis.

$$M_{fi} = -\frac{g^2}{4\pi}\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{i|\tilde{\mathrm{p}}|r\cos\theta'}\frac{e^{-mr}}{r}r^2\sin\theta' d\theta' dr d\phi$$

Integrate over $d\phi$ and set $y = \cos\theta'$.

$$\begin{aligned}
M_{fi} &= -\frac{g^2}{2}\int_0^\infty\int_{-1}^{+1} r e^{i|\tilde{\mathrm{p}}|ry}e^{-mr}dr dy \\
&= -\frac{g^2}{2i|\tilde{\mathrm{p}}|}\int_0^\infty (e^{+i|\tilde{\mathrm{p}}|r} - e^{-i|\tilde{\mathrm{p}}|r})e^{-mr}dr \\
&= -\frac{g^2}{2i|\tilde{\mathrm{p}}|}\int_0^\infty e^{(i|\tilde{\mathrm{p}}|-m)r} - e^{-(i|\tilde{\mathrm{p}}|+m)r}dr \\
&= \frac{g^2}{2i|\tilde{\mathrm{p}}|}\left[\frac{1}{(i|\tilde{\mathrm{p}}| - m)} + \frac{1}{(i|\tilde{\mathrm{p}}| + m)}\right] \\
&= \frac{g^2}{2i|\tilde{\mathrm{p}}|}\left[\frac{2i|\tilde{\mathrm{p}}|}{(-|\tilde{\mathrm{p}}|^2 - m^2)}\right]
\end{aligned}$$

$$M_{fi} = -\frac{g^2}{(m^2+|\vec{\mathrm{p}}|^2)}$$

giving $$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2}\frac{g^4}{(m^2+|\tilde{\mathrm{p}}|^2)^2}$$

★ Scattering in the Yukawa potential introduces a term $(m^2 + |\tilde{\mathrm{p}}|^2)$ in the denominator of the matrix element - this is known a the propagator.

Rutherford Scattering

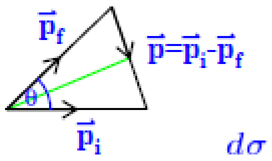
Let $m \rightarrow 0$ and replace $g^2 \rightarrow e^2 = 4\pi\alpha$

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-mr}}{r}$$

gives Coulomb potential: $$V(r) = -\alpha/r$$

Hence for elastic scattering in the Coulomb potential:

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2}\frac{16\pi^2\alpha^2}{|\tilde{\mathrm{p}}|^4}$$

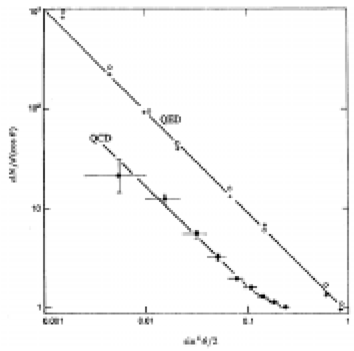


$$|\tilde{\mathrm{p}}| = 2|\tilde{\mathrm{p}}_i|\sin\tfrac{\theta}{2}$$

$$|\tilde{\mathrm{p}}| = 2E\sin\tfrac{\theta}{2}$$

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2}\frac{16\pi^2\alpha^2}{16E^4\sin^4\frac{\theta}{2}}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Rutherford}} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}$$



e.g. The upper points are the Gieger and Marsden data (1911) for the elastic scattering of α particles as they traverse thin gold and silver foils. The scattering rate, plotted versus $\sin^4\frac{\theta}{2}$, follows the Rutherford formula. (note, plotted as log vs. log)

4.1 Decay Rates and Cross Sections

τ Mean life time

Γ Decay rate of particles

Probability of decay per unit time

$$dN(t) = -N(t)\Gamma dt$$

$$N(t) = N(0)e^{-\Gamma t}$$

$$\tau = \frac{1}{\Gamma}$$

Expectation value of time t

4.1 Decay Rates and Cross Sections

In scattering problems, the most interesting quantity is the cross section of the target.

$$n_s = \frac{\#\text{ of scattered particles}}{\text{volume} \times \text{time}}$$

$$\rho_b = \frac{\#\text{ of beam particles}}{\text{volume}}$$

$$v_b = \text{velocity of beam particles}$$

$$\rho_t = \frac{\#\text{ of target particles}}{\text{volume}}$$

$$\Downarrow$$

$$\sigma = \frac{n_s}{\rho_b v_b \rho_t}$$

4.1 Decay Rates and Cross Sections

$$n_s = \frac{2\pi}{\hbar} \left| V_{fi} \right|^2 \rho_f \left(E_f \right)$$

Final state density

$$V_{fi} = \int d^3\vec{x}\,\phi_f^*(\vec{x}) V(\vec{x}) \phi_i(\vec{x})$$

4.1 Decay Rates and Cross Sections

$$\phi(\vec{x}) = \frac{1}{\sqrt{L^3}} e^{i\vec{k}\cdot\vec{x}}$$

Plane wave

Satisfying a kinetic energy equation

Energy conservation for the elastic scattering

$$\int_{L^3} |\phi|^2 d^3\vec{x} = 1$$

$$V_{fi} = \frac{1}{L^3} \int d^3\vec{x} e^{-i\vec{k}_f\cdot\vec{x}} V(\vec{x}) e^{i\vec{k}_i\cdot\vec{x}}$$

4.1 Decay Rates and Cross Sections

$$\rho_f(E) = \frac{mk}{8\pi^3\hbar^2} d\Omega$$

$$\rho_f(E)dE = \frac{d^3\vec{n}}{L^3}$$

$$\vec{n} = \frac{L}{2\pi}\vec{k}$$

Box quantization

$$\rho_f(E)dE = \frac{\left(\frac{L}{2\pi}\right)^3 d^3\vec{k}}{L^3} = \frac{1}{8\pi^3}\left|\vec{k}\right|^2 d\left|\vec{k}\right| d\Omega$$

$$dE = \frac{\hbar^2 k}{m} dk$$

4.1 Decay Rates and Cross Sections

$$\rho_b = \frac{1}{L^3}, \rho_t = \frac{1}{L^3}, v_b = \frac{\hbar k}{m}$$

$$d\sigma = \frac{n_s}{\rho_b v_b \rho_t}$$

$$= \frac{\frac{2\pi}{\hbar}\left(\frac{1}{L^3}\right)^2 \left|\int d^3\vec{x} e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{x}} V(\vec{x})\right|^2 \frac{mk}{8\pi^3\hbar^2} d\Omega}{\frac{1}{L^3}\left(\frac{\hbar k}{m}\right)\frac{1}{L^3}}$$

$$= \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|\int d^3\vec{x} e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{x}} V(\vec{x})\right|^2 d\Omega$$

4.1 Decay Rates and Cross Sections

$$V(\vec{x}) = -\frac{e^2}{|\vec{x}|}$$

Coulomb potential

$$\int d^3\vec{x}\, e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{x}} V(\vec{x})$$

$$= -e^2 \int d^3\vec{x} \frac{e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{x}}}{|\vec{x}|}$$

$$= -\frac{4\pi e^2}{\left|\vec{k}_i - \vec{k}_f\right|^2}$$

4.1 Decay Rates and Cross Sections

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \frac{(4\pi)^2 e^4}{\left|\vec{k}_i - \vec{k}_f\right|^4}$$

$$= \left(\frac{e^2}{4E\sin^2\frac{\theta}{2}}\right)^2$$

$$\left(\vec{k}_i - \vec{k}_f\right)^2 = 2k^2(1-\cos\theta)$$
$$= 4k^2\sin^2\frac{\theta}{2}$$
$$= \frac{8mE}{\hbar^2}\sin^2\frac{\theta}{2}$$

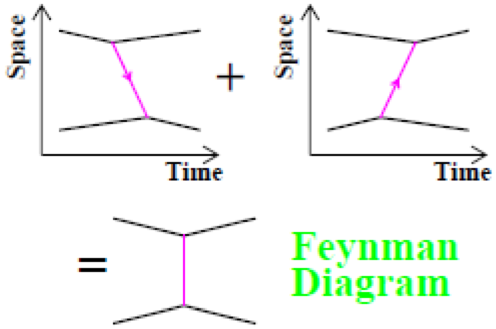
=> Rutherford Scattering

Feynman Diagrams

★ The results of calculations based on a single process in Time-Ordered Perturbation Theory (sometimes called old-fashioned, OFPT) depend on the reference frame.

★ However, the sum of all time orderings is not frame dependent and provides the basis for our relativistic theory of Quantum Mechanics.

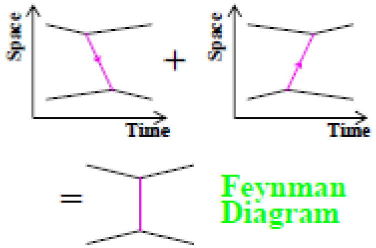
★ The sum of time orderings are represented by FEYNMAN DIAGRAMS



★ Energy and Momentum are conserved at the interaction vertices

★ But the exchanged particle no longer has $m_X^2 = E_X^2 - p_X^2$, it is VIRTUAL

Virtual Particles



Virtual Particles:

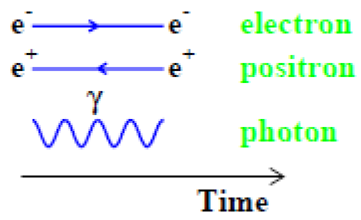
- ★ Forces due to exchanged particle X which is termed VIRTUAL.
- ★ The exchanged particle is off mass-shell, *i.e.* for the unobservable exchanged VIRTUAL particle $E^2 \neq p^2 + m_X^2$.
- ★ *i.e.* $m^2 = E_X^2 - p_X^2$ does not give the physical mass, m_X. The mass of the virtual particle $m^2 = E_X^2 - p_X^2$ can be +ve or -ve.

Qualitatively: the propagator is inversely proportional to how far the particle is off-shell. The further off-shell, the smaller the probability of producing such a virtual state.

Understanding Feynman Diagrams

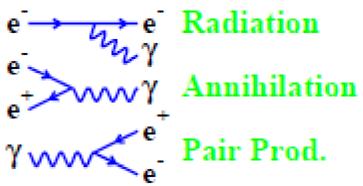
★ Feynman diagrams are the language of modern particle physics. They will be used extensively throughout this course.

The Basic Building Blocks



Note : the positron (e^+) line is drawn as a negative energy electron traveling backwards in time

The $e^\pm$ — photon interactions

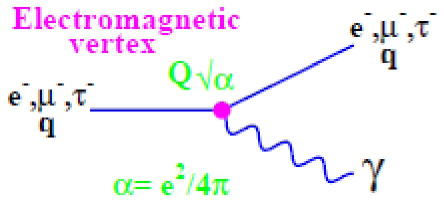


Note: none of these processes are allowed in isolation : Forbidden by $(E, \tilde{\mathrm{p}})$ conservation.

★ The strength of the interaction between the virtual photon and fermions is called the coupling strength. For the electromagnetic interaction this is proportional to electric charge e.

The Electromagnetic Vertex

★ The electromagnetic interaction is described by the photon propagator and the vertex:



COUPLING strength proportional to the fermion charge.

★ All electromagnetic interactions can be described in terms of the above diagram

★ Always conserve energy and momentum + (angular momentum, charge)

★ QED Vertex NEVER changes flavour i.e. $e^- \rightarrow e^-\gamma$ but not $e^- \rightarrow \mu^-\gamma$

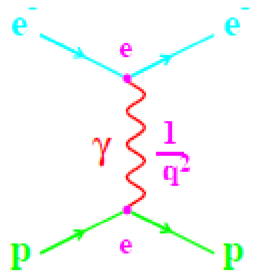
★ QED Vertex also conserves PARITY

★ Qualitatively : $Q\sqrt{\alpha}$ can be thought of the probability of a charged particle emitting a photon, the probability is proportional to $1/q^2$ of the photon.

Physics with Feynman Diagrams

Scattering cross sections calculated from:

- ★ Fermion wave functions
- ★ Vertex Factors : coupling strength
- ★ Propagator
- ★ Phase Space



Electron Current

Propagator

Proton Current

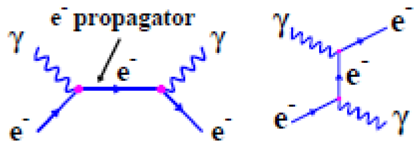
Matrix element M factorises into 3 terms :

$$
\begin{aligned}
-iM &= \langle \overline{u}_e | ie\gamma^\mu | u_e \rangle && \text{Electron Current} \\
&\times \frac{-ig^{\mu\nu}}{q^2} && \text{Photon Propagator} \\
&\times \langle \overline{u}_p | ie\gamma^\nu | u_p \rangle && \text{Proton Current}
\end{aligned}
$$

The factors γ^μ and $g^{\mu\nu}$ are 4×4 matrices which account for the spin-structure of the interaction (described in the lecture on the Dirac Equation).

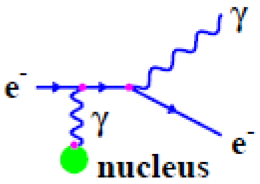
Pure QED Processes

Compton Scattering



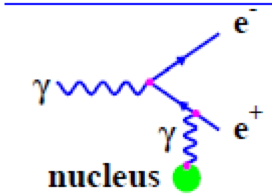
$$M \sim e^2$$
$$\sigma \sim |M|^2 \sim e^4$$
$$\sigma \sim (4\pi)^2\alpha^2$$

Bremsstrahlung



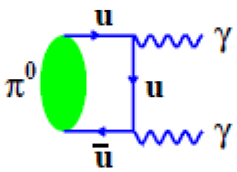
$$M \sim Ze.e.e$$
$$|M|^2 \sim Z^2e^6$$
$$\sigma \sim (4\pi)^3 Z^2\alpha^3$$

e^+e^- Pair Production



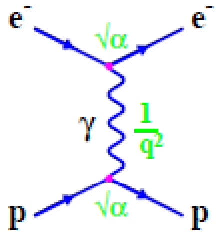
$$M \sim e.e.Ze$$
$$|M|^2 \sim Z^2e^6$$
$$\sigma \sim (4\pi)^3 Z^2\alpha^3$$

π^0 Decay



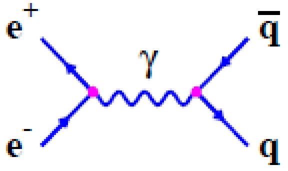
$$M \sim Q_u e.Q_u e$$
$$|M|^2 \sim Q_u^4 e^4$$
$$\sigma \sim (4\pi)^2 Q_u^4\alpha^2$$

Electron-Proton Scattering



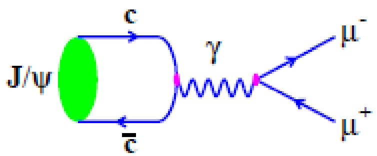
$$M \sim e.e$$
$$|M|^2 \sim e^4$$
$$\sigma \sim (4\pi)^2\alpha^2$$

e^+e^- Annihilation



$$M \sim e.Q_u e$$
$$|M|^2 \sim Q_u^2 e^4$$
$$\sigma \sim (4\pi)^2 Q_u^2\alpha^2$$

$J/\psi \to \mu^+\mu^-$

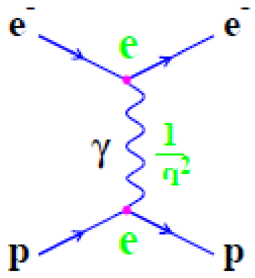


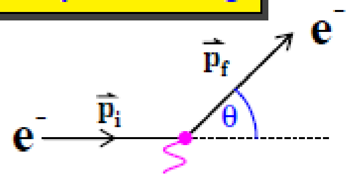
$$M \sim Q_c e.e$$
$$|M|^2 \sim Q_c^2 e^4$$
$$\sigma \sim (4\pi)^2 Q_c^2\alpha^2$$

Coupling strength determines ‘order of magnitude’ of matrix element. For particles interacting/decaying via electromagnetic interaction: typical values for cross sections/lifetimes

$$\sigma_{em} \sim 10^{-2}\ \text{mb}$$
$$\tau_{em} \sim 10^{-20}\ \text{s}$$

"Spin-less" e-p Scattering





$$M = \frac{e^2}{q^2} = \frac{4\pi\alpha}{q^2}$$

From Handout 1, pages 31-34:

$$\frac{d\sigma}{d\Omega} = 2\pi|M|^2\frac{E^2}{(2\pi)^3}$$

$$= 2\pi\frac{(4\pi\alpha)^2}{q^4}\frac{E^2}{(2\pi)^3} = \frac{4\alpha^2E^2}{q^4}$$

q^2 is the four-momentum transfer:

$$\begin{aligned} q^2 &= q^\mu q_\mu = (E_f - E_i)^2 - (\tilde{p}_f - \tilde{p}_i)^2 \\ &= E_f^2 + E_i^2 - 2E_fE_i - \tilde{p}_f^{\,2} - \tilde{p}_i^{\,2} + 2.\tilde{p}_f.\tilde{p}_i \\ &= 2m_e^2 - 2E_fE_i + 2|\tilde{p}_f||\tilde{p}_i|\cos\theta \end{aligned}$$

neglecting electron mass: *i.e.* $m_e^2 = 0$ and $|\tilde{p}_f| = E_f$

$$q^2 = -2E_iE_f(1-\cos\theta)$$

$$q^2 = -4E_iE_f\sin^2\frac{\theta}{2}$$

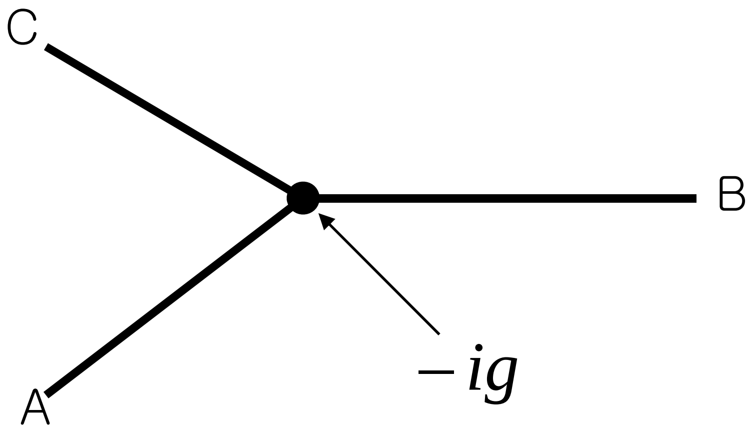
Therefore for ELASTIC scattering $E_i = E_f$

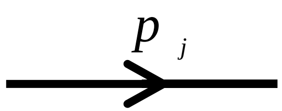
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}$$

i.e. the Rutherford scattering formula (Handout 1 p.36)

4.3 Application of Feynman Rules

Before we start the real interactions it may be useful to train ourselves in using the Feynman rules and deriving cross sections and decay rates.





$$= \frac{i}{p_j^2 - m_j^2 c^2}$$

The Feynman rule of the propagater

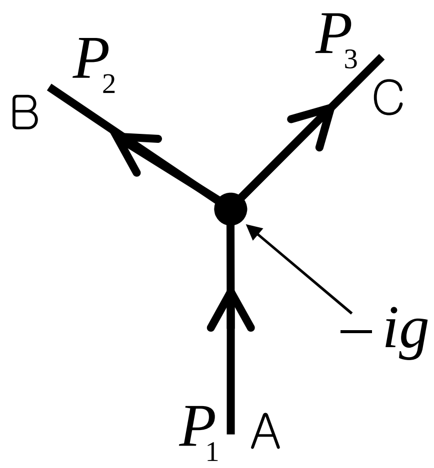
4.3 Application of Feynman Rules

1. Draw all possible diagrams in a given order of coupling constant for the particular process that is under consideration.
2. Assign all the momenta of internal and external lines for each diagram satisfying the conservation of four momenta.
3. Give $-ig$ factor for each vertex and $\frac{i}{p_j^2 - m_j^2 c^2}$ factor for the propagator when the momentum is given by p_j and the mass of the propagating particle is m_j.
4. Combine all the vertex and propagator factors to obtain the invariant amplitude $-i\mathrm{M}$
5. If the diagram includes the loop, integrate over the loop momentum q, i.e.

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4}$$

4.3 Application of Feynman Rules

Consider the decay process $A \rightarrow B + C$



$$-i\mathrm{M} = -ig$$

No propagator!

4.3 Application of Feynman Rules

$$d\Gamma = \frac{g^2}{2\hbar m_A}\left(\frac{c}{(2\pi)^3}\frac{d^3\overrightarrow{p_2}}{2E_2}\right)\left(\frac{c}{(2\pi)^3}\frac{d^3\overrightarrow{p_3}}{2E_3}\right)(2\pi)^4\delta^4(p_1 - p_2 - p_3)$$

or

$$\Gamma = \frac{g^2}{2\hbar m_A}\int\frac{cd^3\overrightarrow{p_2}}{(2\pi)^3 2E_2}\frac{cd^3\overrightarrow{p_3}}{(2\pi)^3 2E_3}(2\pi)^4\delta^4(p_1 - p_2 - p_3)$$

$$\int\frac{cd^3\overrightarrow{p}}{2E} = \int_{-\infty}^{\infty} d^4 p\,\delta\left(p^2 - m^2c^2\right)\theta(p_0)$$

$$\Gamma = \frac{g^2}{2\hbar m_A}\cdot\frac{c}{(2\pi)^2}\int\frac{d^3\overrightarrow{p_2}}{2E_2}\delta(p_3{}^2 - m_C^2c^2)$$

$$d^3\overrightarrow{p_2} = \overrightarrow{p_2}^2 d\left|\overrightarrow{p_2}\right|d\Omega$$

4.3 Application of Feynman Rules

$$\Gamma = \frac{g^2}{2\hbar m_A} \cdot \frac{c}{(2\pi)^2} \cdot 4\pi \int \frac{d\left|\vec{p}_2\right|}{2E_2} \left|\vec{p}_2\right|^2 \delta\left((p_1 - p_2)^2 - m_C^2 c^2 \right)$$

$$= \frac{g^2 c}{4\pi\hbar m_A} \int \frac{d\left|\vec{p}_2\right| \left|\vec{p}_2\right|^2}{E_2} \delta\left(m_A^2 c^2 + m_B^2 c^2 - 2 m_A E_2 - m_C^2 c^2 \right)$$

$$= \frac{g^2}{4\pi\hbar m_A c} \int_0^\infty dE_2 \left|\vec{p}_2\right| \frac{1}{2m_A} \delta\left(E_2 - \frac{m_A^2 + m_B^2 - m_C^2}{2m_A} c^2 \right)$$

$$= \frac{g^2 \left|\vec{p}_2\right|}{8\pi\hbar m_A^2 c}$$

$$E_2 dE_2 = \left|\vec{p}_2\right| d\left|\vec{p}_2\right| c^2$$

4.3 Application of Feynman Rules

$$\left|\vec{p}_2\right| = \sqrt{\frac{E^2 - m_B^2 c^4}{c}}$$

$$= \frac{1}{c}\sqrt{\left(\frac{m_A^2 + m_B^2 - m_C^2}{2m_A}c^2\right)^2 - m_B^2 c^4}$$

$$= \frac{c}{2m_A}\sqrt{\left(m_A^2 + m_B^2 - m_C^2 - 2m_A m_B\right)\left(m_A^2 + m_B^2 - m_C^2 + 2m_A m_B\right)}$$

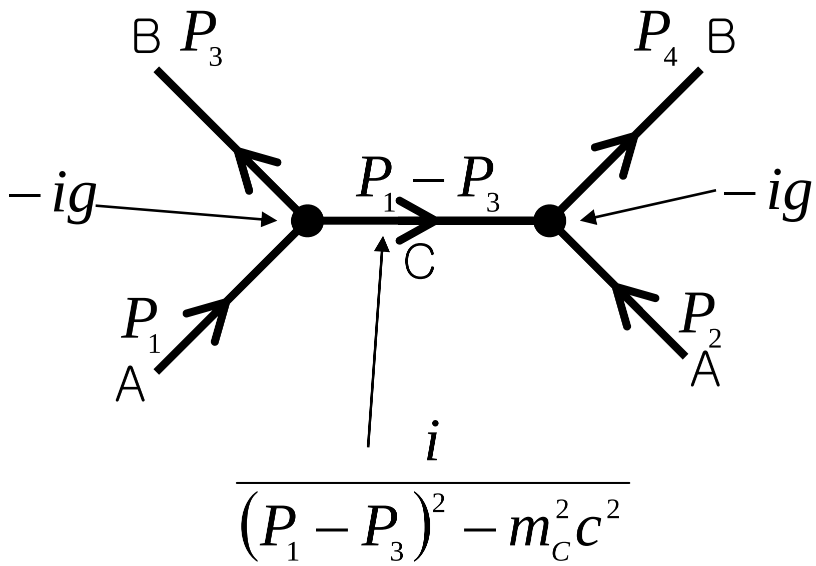
$$= \frac{c}{2m_A}\sqrt{\left(m_A - m_B + m_C\right)\left(m_A - m_B - m_C\right)\left(m_A + m_B + m_C\right)\left(m_A + m_B - m_C\right)}$$

$$\boxed{\tau = \frac{1}{\Gamma} = \frac{8\pi\hbar m_A{}^2 c}{g^2\left|\vec{p}_2\right|}}$$

4.3 Application of Feynman Rules

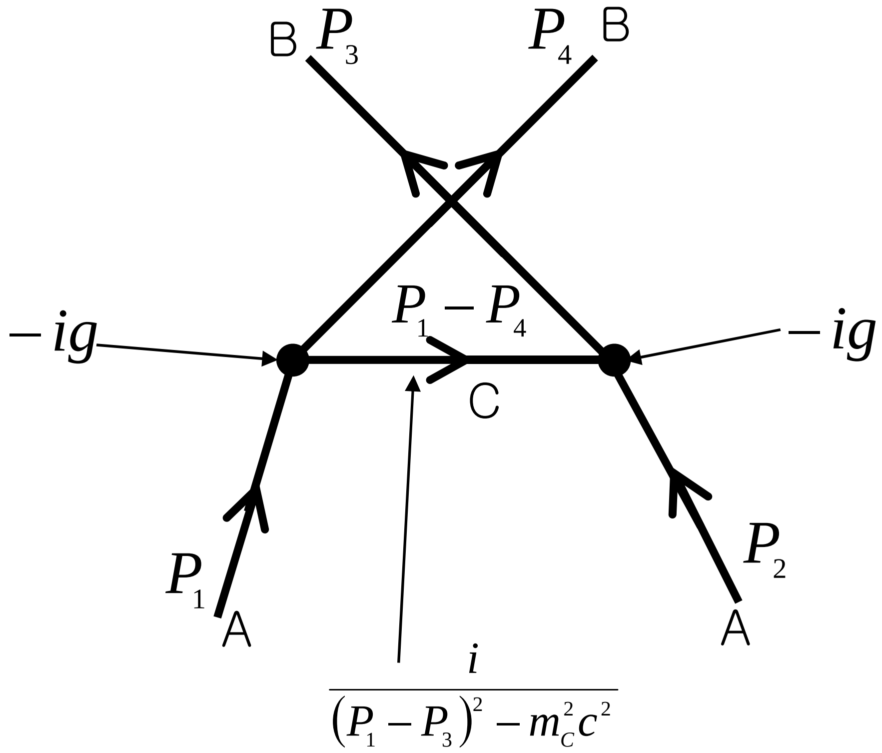
Now, consider a scattering problem.

$-i\mathrm{M}_1 \quad =$



4.3 Application of Feynman Rules

$$-i\mathrm{M}_2 \;=$$



4.3 Application of Feynman Rules

$$\mathrm{M}_1 = \frac{g^2}{(p_1 - p_3)^2 - m_C^2 c^2}$$

$$\mathrm{M}_2 = \frac{g^2}{(p_1 - p_4)^2 - m_C^2 c^2}$$

$$p_1 = \left(\sqrt{\vec{p}^{\,2} + m_A^2 c^2}, \vec{p} \right)$$

$$p_2 = \left(\sqrt{\vec{p}^{\,2} + m_A^2 c^2}, -\vec{p} \right)$$

$$p_3 = \left(\sqrt{\vec{p}'^2 + m_B^2 c^2}, \vec{p}' \right)$$

$$p_4 = \left(\sqrt{\vec{p}'^2 + m_B^2 c^2}, -\vec{p}' \right)$$

← c.m system.

4.3 Application of Feynman Rules

$$(p_1 - p_3)^2 - m_C^2 c^2 = m_A^2 c^2 + m_B^2 c^2 - m_C^2 c^2 - 2\left(\sqrt{\vec{p}^{\,2} + m_A^2 c^2} \sqrt{\vec{p}'^2 + m_B^2 c^2} - \vec{p} \cdot \vec{p}' \right)$$

$$= \left(m_A^2 + m_B^2 - m_C^2\right) c^2 - 2\left(\sqrt{\vec{p}^{\,2} + m_A^2 c^2} \sqrt{\vec{p}'^2 + m_B^2 c^2} - |\vec{p}||\vec{p}'|\cos\theta \right)$$

$$(p_1 - p_4)^2 - m_C^2 c^2 = m_A^2 c^2 + m_B^2 c^2 - m_C^2 c^2 - 2\left(\sqrt{\vec{p}^{\,2} + m_A^2 c^2} \sqrt{\vec{p}'^2 + m_B^2 c^2} + \vec{p} \cdot \vec{p}' \right)$$

$$= \left(m_A^2 + m_B^2 - m_C^2\right) c^2 - 2\left(\sqrt{\vec{p}^{\,2} + m_A^2 c^2} \sqrt{\vec{p}'^2 + m_B^2 c^2} + |\vec{p}||\vec{p}'|\cos\theta \right)$$

θ is the c.m. scattering angle between $\vec{p}$ and $\vec{p}'$

4.3 Application of Feynman Rules

$$d\sigma = |M|^2 \frac{\hbar^2 S}{4\sqrt{\left\{(\vec{p}^{\,2} + m_A^2 c^2) + \vec{p}^{\,2}\right\} - (m_A^2 c^2)^2}} \left(\frac{c}{(2\pi)^3}\frac{d^3\vec{p}_3}{2E_3}\right)\left(\frac{c}{(2\pi)^3}\frac{d^3\vec{p}_4}{2E_4}\right) \times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$= |M|^2 \frac{\hbar^2 S}{8\sqrt{\vec{p}^{\,2}(\vec{p}^{\,2} + m_A^2 c^2)^2}} \left(\frac{c}{(2\pi)^3}\frac{d^3\vec{p}_3}{2E_3}\right)\left(\frac{c}{(2\pi)^3}\frac{d^3\vec{p}_4}{2E_4}\right) \times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$d\sigma = |M|^2 \frac{\hbar^2 S}{8\sqrt{\vec{p}^{\,2}(\vec{p}^{\,2} + m_A^2 c^2)^2}} \frac{c}{(2\pi)^2}\frac{d^3\vec{p}_3}{2E_3} \delta^4(p_4{}^2 - m_B{}^2 c^2)$$

using $d^3\vec{p}_3 = \left|\vec{p}_3\right|^2 d\left|\vec{p}_3\right| d\Omega = \frac{1}{c^2}\left|\vec{p}_3\right| E_3 dE_3 d\Omega$

4.3 Application of Feynman Rules

$$\frac{d\sigma}{d\Omega} = |M|^2 \frac{\hbar^2 S}{8\sqrt{\vec{p}^{\,2}(\vec{p}^{\,2}+m_A^2c^2)^2}} \frac{1}{(2\pi)^2 c} \frac{1}{2} \int dE_3 |\vec{p_3}| \delta\left((p_1+p_2-p_3)^2 - m_B{}^2c^2\right)$$

$$= |M|^2 \frac{\hbar^2 S}{8\sqrt{\vec{p}^{\,2}(\vec{p}^{\,2}+m_A^2c^2)^2}} \frac{1}{8\pi^2 c} \int dE_3 |\vec{p}'| \delta\left(4(\vec{p}^{\,2}+m_A^2c^2)^2 - 4\frac{E_3}{c}\sqrt{\vec{p}^{\,2}+m_A^2c^2}\right)$$

$$= |M|^2 \frac{\hbar^2 S}{8\sqrt{\vec{p}^{\,2}(\vec{p}^{\,2}+m_A^2c^2)^2}} \frac{1}{8\pi^2 c} \frac{c}{4\sqrt{\vec{p}^{\,2}+m_A^2c^2}} \int dE_3 |\vec{p}'| \delta\left(E_3 - c\sqrt{\vec{p}^{\,2}+m_A^2c^2}\right)$$

$$= |M|^2 \frac{\hbar^2 S |\vec{p}'|}{(16\pi)^2(\vec{p}^{\,2}+m_A^2c^2)|\vec{p}|}$$

where $\sqrt{\vec{p}'^2+m_B^2c^2} = \sqrt{\vec{p}^{\,2}+m_B^2c^2}$

4.3 Application of Feynman Rules

$$\frac{d\sigma}{d\Omega}=\frac{1}{2}\cdot\left(\frac{\hbar g^2}{16\pi}\right)^2\cdot\frac{\left|\vec{p}\,'\right|}{(\vec{p}^{\,2}+m_A^2c^2)\left|\vec{p}\right|}$$

$$\left(\frac{1}{\left(m_A^2+m_B^2-m_C^2\right)c^2-2\left(\sqrt{\vec{p}^{\,2}+m_A^2c^2}\sqrt{\vec{p}\,'^{2}+m_B^2c^2}-\left|\vec{p}\right|\left|\vec{p}\,'\right|\cos\theta\right)}\right.$$

$$\left.+\frac{1}{\left(m_A^2+m_B^2-m_C^2\right)c^2-2\left(\sqrt{\vec{p}^{\,2}+m_A^2c^2}\sqrt{\vec{p}\,'^{2}+m_B^2c^2}+\left|\vec{p}\right|\left|\vec{p}\,'\right|\cos\theta\right)}\right)^2$$

where $\left|\vec{p}\,'\right|=\sqrt{\vec{p}^{\,2}+\left(m_A^2-m_B^2\right)c^2}$ and the statistical factor $S=1/2$

4.3 Application of Feynman Rules

In the zero mass limit, $m_A = m_B = m_C = 0$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \cdot \left(\frac{\hbar g^2}{16\pi \left|\vec{p}\right|^3 \sin^2\theta} \right)^2$$

References

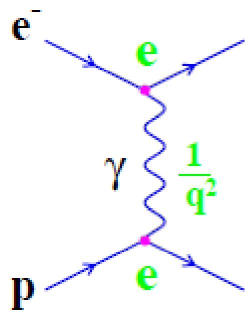
- 윤석훈
- M.A. Thomson
- Richard Kass

Quantum Electrodynamics

QUANTUM ELECTRODYNAMICS: is the quantum theory of the electromagnetic interaction.

CLASSICAL PICTURE: Action at a distance : forces arise from $\tilde{E}$ and $\tilde{B}$ fields. Particles act as sources of the fields $\rightarrow V(\tilde{r})$.

Q.E.D. PICTURE: Forces arise from the exchange of virtual field quanta.



Although a complete derivation of the theory of Q.E.D. and Feynman diagrams is beyond the scope of this course, the main features will be derived.

Interaction via Particle Exchange

NON-EXAMINABLE

FERMI'S GOLDEN RULE for Transition rate, Γ_{fi}:

$$\Gamma_{fi} = \frac{2\pi}{\hbar}|M_{fi}|^2\rho(E_f)$$

$\rho(E_f)$ = density of final states.

★ From 1^{st} order perturbation theory, matrix element M_{fi}:

$$\mathrm{M}_{fi} = \langle\psi_f|\hat{\mathrm{H}}'|\psi_i\rangle$$

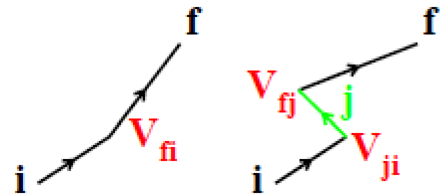
where $\hat{\mathrm{H}}'$ is the operator corresponding to the perturbation to the Hamiltonian.

★ This is only the 1^{st} order term in the perturbation expansion. In 2^{nd} order perturbation theory:

$$\mathrm{M}_{fi} \rightarrow \mathrm{M}_{fi} + \sum_{j\neq i}|\mathrm{M}_{fj}|\frac{1}{E_i - E_j}|\mathrm{M}_{ji}|$$

where the sum is over all intermediate states j, and E_i and E_j are the energies of the initial and intermediate state

★ For scattering, the 1^{st} and 2^{nd} order terms can be viewed as:

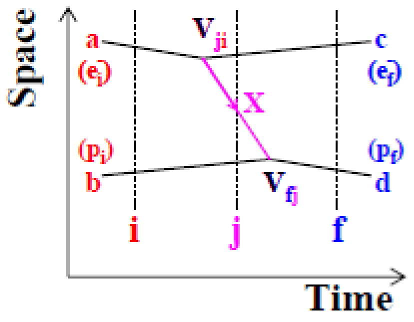


Consider the particle interaction

$$a + b \rightarrow c + d$$

which involves the exchange a particle X. This could be the elastic scattering of electrons and protons, *e.g.* $e^- p \rightarrow e^- p$ where X is an exchanged photon.

★ One possible space-time picture for this process is



Initial State, i: a+b
Final State, f: c+d
Intermediate State, j: b+c+X

★ The Time Ordered interaction consists of $a \rightarrow c + X$ followed by $b + X \rightarrow d$. For example $e_i^- p_i \rightarrow e_f^- p_f$ has the electron emitting a photon $(e_i^- \rightarrow e_f^- \gamma)$ followed by the photon being absorbed by the proton $(p_i \gamma \rightarrow p_f)$.

★ The corresponding term in 2^{nd} order PT:

$$\begin{aligned} \mathrm{M}_{fi}^{ab} &= \frac{\langle \psi_f | \hat{\mathrm{H}}' | \psi_j \rangle \langle \psi_j | \hat{\mathrm{H}}' | \psi_i \rangle}{E_i - E_j} \\ &= \frac{\langle \psi_d | \hat{\mathrm{H}}' | \psi_X \psi_b \rangle \langle \psi_c \psi_X | \hat{\mathrm{H}}' | \psi_a \rangle}{(E_a + E_b) - (E_c + E_X + E_b)} \\ &= \frac{\langle \psi_d | \hat{\mathrm{H}}' | \psi_X \psi_b \rangle \langle \psi_c \psi_X | \hat{\mathrm{H}}' | \psi_a \rangle}{(E_a - E_c - E_X)} \end{aligned}$$

Before we go any further some comments:

★ The superscript ab on M_{fi}^{ab} indicates the time ordering where a interacts with X before b

consequently the results are not Lorentz Invariant

i.e. depend on rest frame.

★ Momentum is conserved in $a \to c + X$ and $b + X \to d$.

★ The exchanged particle X is ON MASS SHELL:

$$E_X^2 - p_X^2 \quad = \quad m_X^2$$

★ The matrix elements $\langle\psi_d|\hat{\mathrm{H}}'|\psi_X\psi_b\rangle$ and $\langle\psi_c\psi_X|\hat{\mathrm{H}}'|\psi_a\rangle$ depend on the "strength" of the interaction. *e.g.* the strength of the γe^- and γp interaction which determines the probability that an electron(proton) will emit(absorb) a photon.

★ For the electromagnetic interaction:

$$\langle\psi_k|\hat{\mathrm{H}}'|\psi_j\rangle = e\epsilon_0\langle\psi_k|z|\psi_j\rangle$$

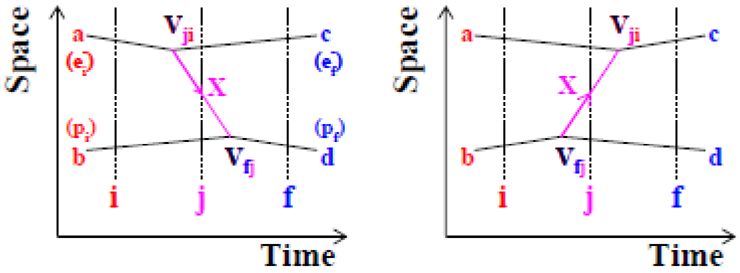
for a photon with polarization in the z-direction. (see Dr Ritchie's QM II lecture 10)

★ Neglecting spin (*i.e.* for assuming all particles are spin-0 *i.e.* scalars) the ME becomes:

$$\langle\psi_d|\hat{\mathrm{H}}'|\psi_X\psi_b\rangle = e$$

★ More generally, $\langle\psi_d|\hat{\mathrm{H}}'|\psi_X\psi_b\rangle = g$, where g is the interaction strength.

Now consider the <u>other</u> time ordering $b \rightarrow d + X$ followed by $a + X \rightarrow b$



The corresponding term in 2^{nd} order PT:

$$\mathrm{M}_{fi}^{ba} = \frac{\langle \psi_c|\hat{\mathrm{H}}'|\psi_X\psi_a\rangle\langle\psi_d\psi_X|\hat{\mathrm{H}}'|\psi_b\rangle}{(E_a+E_b)-(E_d+E_X+E_a)}$$
$$= \frac{\langle \psi_c|\hat{\mathrm{H}}'|\psi_X\psi_a\rangle\langle\psi_d\psi_X|\hat{\mathrm{H}}'|\psi_b\rangle}{(E_b-E_d-E_X)}$$
$$= \frac{\langle \psi_c|\hat{\mathrm{H}}'|\psi_X\psi_a\rangle\langle\psi_d\psi_X|\hat{\mathrm{H}}'|\psi_b\rangle}{(E_b-E_d-E_X)}$$

Assume a common interaction strength, g, at both vertices,

i.e. $\langle \psi_c|\hat{\mathrm{H}}'|\psi_X\psi_a\rangle = \langle\psi_d\psi_X|\hat{\mathrm{H}}'|\psi_b\rangle = g$

$$\Rightarrow \quad \mathrm{M}_{fi}^{ba} = \frac{g^2}{(E_b-E_d-E_X)} \times \frac{1}{2E_X}$$

WARNING : I have introduced an (unjustified) factor of $\frac{1}{2E_X}$. This arises from the relativistic normalization of the wave-function for particle X (see appendix). For initial/final state particles the normalisation is cancelled by corresponding terms in the flux/phase-space. For the "intermediate" particle X no such cancelation occurs.

Now sum over two time ordered transition rates

$$M_{fi} = M_{fi}^{ab} + M_{fi}^{ba}$$
$$= g^2 \left(\frac{1}{E_a\text{-}E_c\text{-}E_X} + \frac{1}{E_b\text{-}E_d\text{-}E_X} \right) \times \frac{1}{2E_X}$$

since
$$E_a + E_b = E_c + E_d$$
$$\Rightarrow E_b - E_d = E_c - E_a$$

giving:

$$M_{fi} = g^2 \left(\frac{1}{E_a\text{-}E_c\text{-}E_X} + \frac{1}{E_c\text{-}E_a\text{-}E_X} \right) \times \frac{1}{2E_X}$$
$$= g^2 \left(\frac{1}{E_a\text{-}E_c\text{-}E_X} - \frac{1}{E_a\text{-}E_c\text{+}E_X} \right) \times \frac{1}{2E_X}$$
$$= g^2 \frac{2E_X}{(E_a - E_c)^2 - E_X^2} \times \frac{1}{2E_X}$$

From the first time ordering:

$$E_X^2 = (\tilde{p}_a - \tilde{p}_c)^2 + m_X^2$$

therefore

$$M_{fi} = \frac{g^2}{(E_a - E_c)^2 - (\tilde{p}_a - \tilde{p}_c)^2 - m_X^2}$$
$$M_{fi} = \frac{g^2}{q^2 - m_X^2}$$

with $q^2 = q^\mu q_\mu = E^2 - |\tilde{p}|^2$

where $(E, |\tilde{p}|)$ are energy/momentum carried by the virtual particle. The **SUM** of time-ordered processes depends on q^2 and is therefore Lorentz invariant ! The 'invariant mass' of the exchanged particle, X, $m_{inv}^2 = E^2 - |\tilde{p}|^2$, is NOT the **REST MASS**, m_X.

The term $$\frac{1}{q^2 - m^2}$$

is called the PROPAGATOR

It corresponds to the term in the matrix element arising from the exchange of a massive particle which mediates the force. For massless particles e.g. photons :

$$\frac{1}{q^2}$$

NOTE: q^2 is the 4-momentum of the exchanged particle ($q^2 = q^\mu q_\mu = E^2 - |\tilde{\mathrm{p}}|^2$)

Previously (page 35 of HANDOUT 1) we obtained the matrix element for elastic scattering in the YUKAWA potential:

$$M_{fi}^{YUK} = -\frac{g^2}{(m^2 + |\vec{\mathrm{p}}|^2)}$$

For elastic scattering $E_X = 0$, and $q^2 = -|\tilde{\mathrm{p}}|^2$

$$M_{fi}^{YUK} \rightarrow \frac{g^2}{q^2 - m^2}$$

Which is exactly the expression obtained on the previous page. Hence, elastic scattering via particle exchange in 2nd order P.T. is equivalent to scattering in a Yukawa potential using 1st order P.T.

Action at a Distance

NEWTON : "...that one body can act upon another at a distance, through a vacuum, without the mediation of anything else,..., is to me a great absurdity"

- ★ In Classical Mechanics and non-relativistic Quantum Mechanics forces arise from potentials $V(\tilde{r})$ which act instantaneously over all space.
- ★ In Quantum Field theory, forces are mediated by the exchange of virtual field quanta - and there is no mysterious action at a distance.
- ★ Matter and Force described by 'particles'