Quantum Mechanics

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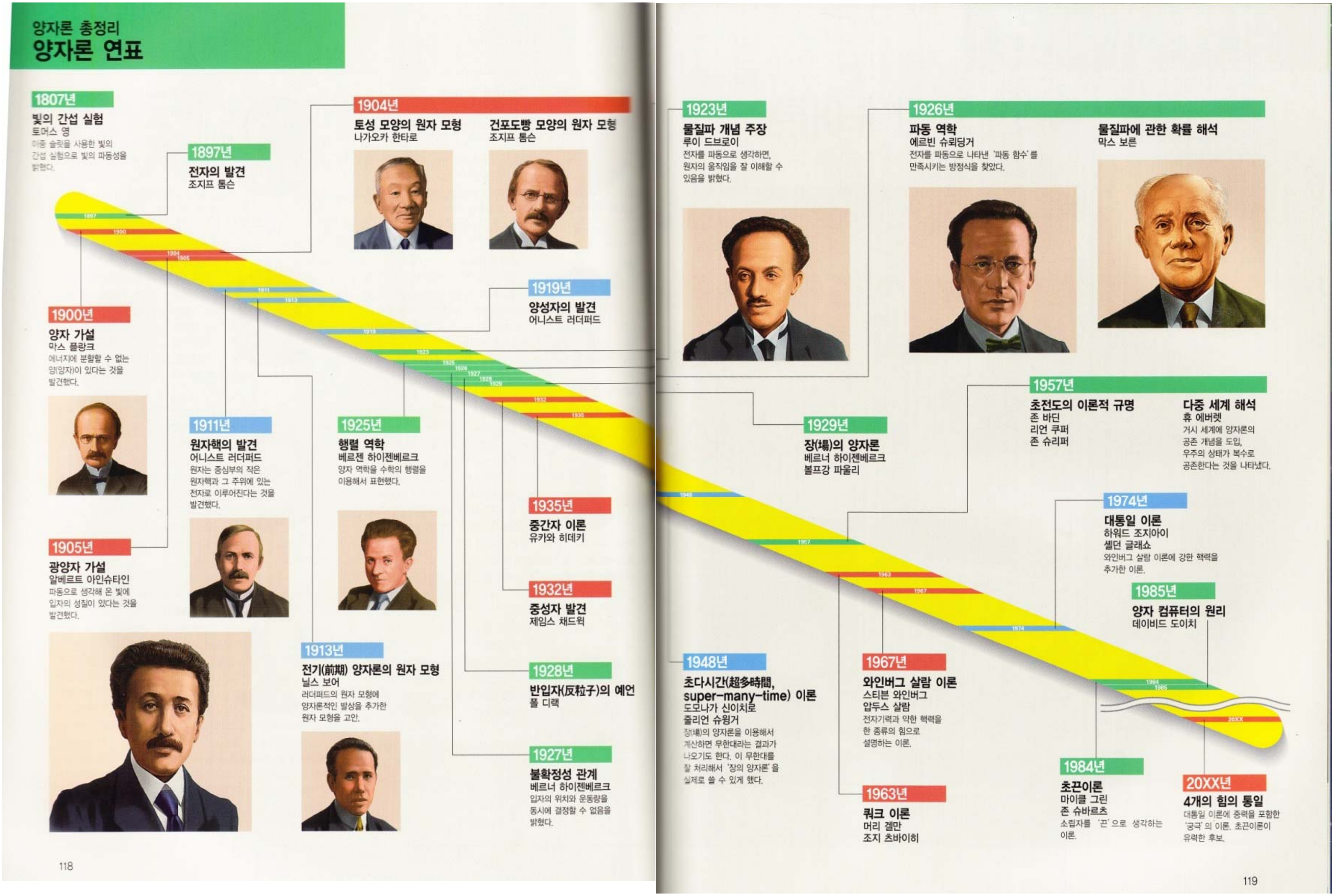
Syllabus

- Introduction (Chap. 1)
- Special Relativity (Chap. 2)
- Quantum Mechanics (Chap. 3)
  - Hydrogen Atom
  - Meson wave function
  - Baryon wave function
  - Magnetic moments
- Detector
- Data Processing
- Feynman diagram (Chap. 4)
- QED (Chap. 5)
- QCD (Chap. 6)
- Weak interaction (Chap. 7)

In this chapter....

- Symmetry properties of the quantum mechanical wave-function space
- Apply it to the Hydrogen atom
- Connection between the atomic system and the hadron system in particle physics
- Meson wave function
- Baryon wave function

양자론 연표



양자론 총정리
양자론 연표

1807년
빛의 간섭 실험
토머스 영
이중 슬릿을 사용한 빛의 간섭 실험으로 빛의 파동성을 밝혔다.

1897년
전자의 발견
조지프 톰슨

1900년
양자 가설
막스 플랑크
에너지에 분할할 수 없는 양(양자)이 있다는 것을 발견했다.

1904년
토성 모양의 원자 모형
나가오카 한타로
건포도빵 모양의 원자 모형
조지프 톰슨

1905년
광양자 가설
알베르트 아인슈타인
파동으로 생각해 온 빛에 입자의 성질이 있다는 것을 발견했다.

1911년
원자핵의 발견
어니스트 러더퍼드
원자는 중심부의 작은 원자핵과 그 주위에 있는 전자로 이루어진다는 것을 발견했다.

1913년
전기(前期) 양자론의 원자 모형
닐스 보어
러더퍼드의 원자 모형에 양자론적인 발상을 추가한 원자 모형을 고안.

1919년
양성자의 발견
어니스트 러더퍼드

1925년
행렬 역학
베르젠 하이젠베르크
양자 역학을 수학의 행렬을 이용해서 표현했다.

1935년
중간자 이론
유카와 히데키

1932년
중성자 발견
제임스 채드윅

1928년
반입자(反粒子)의 예언
폴 디랙

1927년
불확정성 관계
베르너 하이젠베르크
입자의 위치와 운동량을 동시에 결정할 수 없음을 밝혔다.

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1923년
물질파 개념 주장
루이 드브로이
전자를 파동으로 생각하면, 원자의 움직임을 잘 이해할 수 있음을 밝혔다.

1926년
파동 역학
에르빈 슈뢰딩거
전자를 파동으로 나타낸 '파동 함수'를 만족시키는 방정식을 찾았다.
물질파에 관한 확률 해석
막스 보른

1929년
장(場)의 양자론
베르너 하이젠베르크
볼프강 파울리

1957년
초전도의 이론적 규명
존 바딘
리언 쿠퍼
존 슈리퍼
다중 세계 해석
휴 에버렛
거시 세계에 양자론의 공존 개념을 도입, 우주의 상태가 복수로 공존한다는 것을 나타냈다.

1974년
대통일 이론
하워드 조지아이
셀던 글래쇼
와인버그 살람 이론에 강한 핵력을 추가한 이론.

1985년
양자 컴퓨터의 원리
데이비드 도이치

1948년
초다시간(超多時間, super-many-time) 이론
도모나가 신이치로
줄리언 슈윙거
장(場)의 양자론을 이용해서 계산하면 무한대라는 결과가 나오기도 한다. 이 무한대를 잘 처리해서 '장의 양자론'을 실제로 쓸 수 있게 했다.

1967년
와인버그 살람 이론
스티븐 와인버그
압두스 살람
전자기력과 약한 핵력을 한 종류의 힘으로 설명하는 이론.

1963년
쿼크 이론
머리 겔만
조지 츠바이히

1984년
초끈이론
마이클 그린
존 슈바르츠
소립자를 '끈'으로 생각하는 이론.

20XX년
4개의 힘의 통일
대통일 이론에 중력을 포함한 '궁극'의 이론. 초끈이론이 유력한 후보.

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3.1 Angular Momentum and CG-Coefficients

$\vec{J}$ The angular momentum operators

$|jm>$ The quantum state

$$J_z|jm> = m\hbar|jm>$$

Where m = −j,⋯,0,⋯,j.

$$J_z = \frac{\hbar\partial}{i\partial\phi}$$

3.1 Angular Momentum and CG-Coefficients

$$J_{\pm} = J_x \pm iJ_y$$

$$[J_z, J_+] = \hbar J_+$$
$$[J_z, J_-] = -\hbar J_-$$
$$[J_+, J_-] = 2\hbar J_z$$

$$J_+ \left| jm > = \sqrt{j(j+1) - m(m+1)}\hbar \right| jm+1 >$$

$$J_- \left| jm > = \sqrt{j(j+1) - m(m-1)}\hbar \right| jm-1 >$$

$$J^2 \left| jm > = j(j+1)\hbar^2 \right| jm+1 >$$

$$J^2 = \frac{J_+J_- + J_-J_+}{2} + J_z{}^2$$

3.1 Angular Momentum and CG-Coefficients

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

$$\left| jm > \right. = \sum_{j_1 j_2} C^{j j_1 j_2}_{m m_1 m_2} \left| j_1 m_1 > \right| j_2 m_2 > \quad (\text{where } \mathrm{m} = \mathrm{m}_1 + \mathrm{m}_2)$$

Clebsch-Gordon(CG)
coefficient

$$\left| j_1 m_1 > \right| j_2 m_2 > = \sum_{j=|j_1 - j_2|}^{j_1 + j_2} C^{j j_1 j_2}_{m m_1 m_2} \left| jm > \right.$$

SPIN

QM Revision

Quantum Mechanical LADDER operators, $\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y$

$$\begin{aligned} \hat{J}_{+}|j,m\rangle &= \sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle \\ \hat{J}_{-}|j,m\rangle &= \sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle \end{aligned}$$

For Example - can generate all $|j,m\rangle$ states from $|j,j\rangle$

$$|1,1\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = \uparrow\uparrow$$

Since $\hat{J}_x$ etc. formed from derivatives use product rule, d(uv) = udv+vdu, when acting on product of states:

$$\begin{aligned} \hat{J}_{-}|1,1\rangle &= (\hat{J}_{-}|\tfrac{1}{2},\tfrac{1}{2}\rangle)|\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},\tfrac{1}{2}\rangle(\hat{J}_{-}|\tfrac{1}{2},\tfrac{1}{2}\rangle) \\ \sqrt{2}|1,0\rangle &= |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle \\ |1,0\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \end{aligned}$$

For the combination of two spin half particle:

$$\begin{aligned} |1,1\rangle &= \uparrow\uparrow \\ |1,0\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1,-1\rangle &= \downarrow\downarrow \end{aligned}$$

All SYMMETRIC under interchange of $1 \leftrightarrow 2$

Also an orthogonal combination which is ANTI-SYMMETRIC under under interchange of $1 \leftrightarrow 2$

$$|0,0\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

3.1 Angular Momentum and CG-Coefficients

Let's consider the spin angular momentum of a meson.

1 Quark
1 Antiquark

$j_1 = \frac{1}{2}$ $j_2 = \frac{1}{2}$

$$j = |j_1 - j_2|, \cdots, j_1 + j_2$$
$$= 0, 1$$

pseudoscalar

Vector mesons

3.1 Angular Momentum and CG-Coefficients

$$\left|1+1\right> = \left|\frac{1}{2}+\frac{1}{2}\right>\left|\frac{1}{2}+\frac{1}{2}\right>$$

$$m = m_1 + m_2$$

In order to obtain $\left|10\right>$ *state, one can apply the lowering operator.*

$$J_- = J_{1-} + J_{2-}$$

$$\left|10\right> = \frac{1}{\sqrt{2}}\left|\frac{1}{2}-\frac{1}{2}\right>\left|\frac{1}{2}+\frac{1}{2}\right> + \frac{1}{\sqrt{2}}\left|\frac{1}{2}+\frac{1}{2}\right>\left|\frac{1}{2}-\frac{1}{2}\right>$$

$$\left|1-1\right> = \left|\frac{1}{2}-\frac{1}{2}\right>\left|\frac{1}{2}-\frac{1}{2}\right>$$

3.1 Angular Momentum and CG-Coefficients

Let's consider the pseudoscalar meson.

$$\left|00\right> = a\left|\frac{1}{2}+\frac{1}{2}\right>\left|\frac{1}{2}-\frac{1}{2}\right> + b\left|\frac{1}{2}-\frac{1}{2}\right>\left|\frac{1}{2}+\frac{1}{2}\right>$$

$$J_+\left|00\right> = 0$$

$$\Downarrow$$

$$\left|00\right> = \frac{1}{\sqrt{2}}\left|\frac{1}{2}+\frac{1}{2}\right>\left|\frac{1}{2}-\frac{1}{2}\right> - \frac{1}{\sqrt{2}}\left|\frac{1}{2}-\frac{1}{2}\right>\left|\frac{1}{2}+\frac{1}{2}\right>$$

3.2 Central Potential in Quantum Mechanics

Review the main idea of solving the central potential problem.
First, obtaining the nonrelativistic solutions.

$$H = \frac{\vec{p}^{\,2}}{2m} + V(r)$$

$$\frac{\vec{p}^{\,2}}{2m} = \frac{P_r^{\,2}}{2m} + \frac{L^2}{2I}$$

Using spherical coordinate system.

3.2 Central Potential in Quantum Mechanics

$$P_r = m\dot{r} \qquad L = mr^2\dot{\theta} \qquad I = mr^2$$

$$\{\frac{P_r^{\ 2}}{2m} + \frac{L^2}{2mr^2} + V(r)\}\psi(r,\theta,\varphi) = E\Psi(r,\theta,\varphi)$$

$$\psi(r,\theta,\varphi) = R(r)Y(\theta,\varphi)$$

Separation of the variables

3.2 Central Potential in Quantum Mechanics

$$-\frac{2mr^2}{R}\left(\frac{P_r^{\ 2}}{2m}+V(r)-E\right)R=\frac{1}{Y}L^2Y$$

Angular part,

$$L^2Y=(const)Y$$

$$L^2Y_l^m(\theta,\varphi)=l(l+1)\hbar^2Y_l^m(\theta,\varphi)$$

$$L_zY_l^m(\theta,\varphi)=m\hbar Y_l^m(\theta,\varphi)$$

3.2 Central Potential in Quantum Mechanics

$$(\frac{P_r^{\ 2}}{2m} + V(r) - E)R = -\frac{(const)}{2mr^2}R$$

$$\Downarrow$$

$$(\frac{P_r^{\ 2}}{2m} + \frac{l(l+1)\hbar^2}{2mr^2} + V(r))R_l(r) = ER_l(r)$$

3.2 Central Potential in Quantum Mechanics

$$P_r = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$$
$$= \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r$$

$$\Rightarrow \quad P_r^2 = -\hbar^2 \frac{1}{r}\frac{\partial^2}{\partial r^2} r$$

$$-\frac{\hbar^2}{2m}\frac{d^2 U_l(r)}{dr^2} + \left(V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right) U_l(r) = E U_l(r)$$

$$\boxed{U_l(r) = r R_l(r)}$$

3.2 Central Potential in Quantum Mechanics

Consider the Coulomb potential in the hydrogen atom.

$$V(r) = -\frac{e^2}{r}$$

$$\Downarrow$$

$$\frac{d^2 U_l(r)}{dr^2} + \left(\frac{2mE}{\hbar^2} + \frac{2m}{\hbar^2}\frac{e^2}{r} - \frac{l(l+1)}{r^2} \right) U_l(r) = 0$$

3.2 Central Potential in Quantum Mechanics

$$x \equiv 2\kappa r \qquad \kappa = \sqrt{-\frac{2mE}{\hbar^2}} \qquad \nu = \frac{1}{\kappa a}$$

Bohr radius.

$$\left(\frac{d^2}{dx^2} - \frac{l(l+1)}{x^2} + \frac{\nu}{x} - \frac{1}{4}\right) U_l(x) = 0$$

$$U_l(x) = x^{l+1} e^{-\frac{x}{2}} v_l(x)$$

3.2 Central Potential in Quantum Mechanics

$$v_l(x) = \sum_{p=0}^{\infty} \frac{\Gamma(l+1+p-\nu)}{\Gamma(l+1-\nu)} \frac{(2l+1)!}{(2l+1+p)!} \frac{x^p}{p!}$$

The series should be terminated at a finite natural number

$$a_{p+1} = \frac{l+1+p-\nu}{(2l+2+p)(p+1)} a_p$$

$$a_p = \frac{\Gamma(l+1+p-\nu)}{\Gamma(l+1-\nu)} \frac{(2l+1)!}{(2l+1+p)!} \frac{1}{p!}$$

3.2 Central Potential in Quantum Mechanics

$$a_{n'+1} = 0$$

$$l + 1 + n' - \nu = 0$$

$$\Downarrow$$

$$\nu = l + 1 + n'$$

Natural number!

3.2 Central Potential in Quantum Mechanics

$$E_n = -\frac{me^4}{2\hbar^2 n^2}$$

$$\boxed{\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137.036}}$$

$$E_n = -\alpha^2 mc^2 \left(\frac{1}{2n^2}\right)$$

Fine structure constant

$$\approx -13.6\text{eV}/\text{n}^2$$

3.2 Central Potential in Quantum Mechanics

$$n' = n - l - 1 \geq 0$$

$$\Downarrow$$

$$n - 1 \geq l$$

$$-l \leq m \leq +l$$

3.3 H-atom Spectra

1. Relativistic Correction

$$T = (\gamma - 1)mc^2$$

$$= \frac{1}{2}m\vec{v}^2 + \frac{3}{8}m\frac{\vec{v}^2}{c^4} + \cdots$$

$$= \frac{\vec{\mathrm{p}}^2}{2m} - \frac{\vec{p}^4}{8m^3c^2} + \cdots$$

$$\boxed{E = \sqrt{\vec{p}^2c^2 + m^2c^4}}$$

3.3 H-atom Spectra

$$\Delta \mathrm{H}_{\mathrm{rel}} = -\frac{\vec{p}^{\,4}}{8m^3c^2}$$

$$\boxed{\frac{\vec{\mathrm{p}}^{\,2}}{2m} = E_n - V}$$

$$\left\langle \left(\frac{\vec{p}^{\,2}}{2m} \right)^2 \right\rangle = \left\langle \left(\frac{\vec{p}^{\,4}}{4m^2} \right) \right\rangle = E_n^2 - 2E_n \langle V \rangle + \langle V^2 \rangle$$

$$\langle \Delta H_{rel} \rangle = -\frac{1}{2mc^2}\left(E_n^2 - 2E_n \langle V \rangle + \langle V^2 \rangle \right)$$

$$\boxed{V = -\frac{e^2}{r}}$$

3.3 H-atom Spectra

$$\boxed{\left\langle \frac{1}{r} \right\rangle = \frac{1}{n^2 a}} \qquad \boxed{\left\langle \frac{1}{r^2} \right\rangle = \frac{1}{(l+\frac{1}{2})n^3 a^2}}$$

$$\Downarrow$$

$$\Delta E_{rel} = -\alpha^2 mc^2 \frac{1}{4n^4}\left(\frac{2n}{l+\frac{1}{2}} - \frac{3}{2} \right)$$

3.3 H-atom Spectra

2. Spin-Orbit Coupling

$$\Delta H_{so} = -\vec{\mu} \cdot \vec{B}$$

$$\boxed{\vec{\mu} = -\frac{e}{mc}\vec{s}}$$

$$\boxed{\vec{B} = \frac{e}{mcr^3}\vec{L}}$$

g-factor of electron is 2

3.3 H-atom Spectra

$$\Delta H_{so} = \frac{e^2}{m^2 c^2 r^3} \vec{L} \cdot \vec{s}$$

$$\Downarrow$$

Correction including Thomas-precession

$$\Delta H_{so} = \frac{e^2}{2m^2 c^2 r^3} \vec{L} \cdot \vec{s}$$

3.3 H-atom Spectra

$$\left\langle \Delta H_{so} \right\rangle = \frac{e^2}{2m^2c^2} \left\langle \frac{\vec{L} \cdot \vec{s}}{r^3} \right\rangle$$

$$= \frac{e^2}{2m^2c^2} \left\langle \frac{\left(\vec{J}^2 - \vec{L}^2 - \vec{s}^2 \right) / 2}{r^3} \right\rangle$$

$$= \frac{e^2}{2m^2c^2} \cdot \frac{\hbar^2 \{ j(j+1) - l(l+1) - \frac{3}{4} \}}{2} \left\langle \frac{1}{r^3} \right\rangle$$

3.3 H-atom Spectra

$$\text{Using } \left\langle \frac{1}{r^3} \right\rangle = \frac{1}{l(l+\frac{1}{2})(l+1)n^3a^3}$$

$$\langle \Delta E_{so} \rangle = \alpha^4 mc^2 \frac{\{j(j+1) - l(l+1) - \frac{3}{4}\}}{4n^3 l(l+\frac{1}{2})(l+1)}$$

3.3 H-atom Spectra

3. The Combination of Relativistic and Spin-Orbit Correction

$$\Delta E_{fs} = \Delta E_{rel} + \Delta E_{so}$$

$$= -\alpha^4 mc^2 \frac{1}{4n^4}\left(\frac{2n}{j+\frac{1}{2}} - \frac{3}{2}\right)$$

$\boxed{Not\ \ depend\ \ on\ \ \ell}$

3.4 Quarkonium Spectra

$$\Delta H_{hf} = -\vec{\mu}_p \cdot (\vec{B}_L + \vec{B}_s)$$

Hyperfine: Proton과 electron의
orbital 운동에 따른 자기장 및 스핀

$$\boxed{\vec{\mu}_p = \gamma_p \frac{e}{Mc} \vec{s}_p}$$

$$\boxed{\vec{B}_L = -\frac{e}{mcr^3} \vec{L}}$$

$$\boxed{\vec{B}_s = \frac{1}{r^3} \{3 \frac{\left(\vec{u} \cdot \vec{r}\right)\vec{r}}{r^2} - \vec{\mu}\}}$$

3.4 Quarkonium Spectra

$$\Delta E_{hf} = \left\langle \Delta H_{hf} \right\rangle \approx \left(\frac{m}{M} \right) \alpha^4 mc^2$$

In the positronium system, this factor is equal to 1

<u>So, there is no distinction between the fine structure and the hyperfine structure in the positronium system.</u>

3.4 Quarkonium Spectra

There are two important aspect of positronium spectra.

1. The reduced mass effect

$$\boxed{m_{red} = \frac{m}{2}}$$

3.4 Quarkonium Spectra

$$\boxed{\begin{aligned} E_n^{positronium} &= -\alpha^2 m_{red} c^2 \left(\frac{1}{2n^2} \right) \\ &= \frac{1}{2} E_n^{hydrogen} \\ &\approx \frac{13.6eV}{2} = 6.8eV \end{aligned}}$$

$$\boxed{\begin{aligned} a^{positronium} &= \frac{\hbar^2}{m_{red} e^2} \\ &= 2a^{hydrogen} \\ &= 1.06 \times 10^{-8}\, cm \end{aligned}}$$

3.4 Quarkonium Spectra

__2. The electron-positron annihilation effect__
which modifies the interaction Hamiltonian.

$$\Delta E_{annihilation} \approx \alpha^4 mc^2$$

3.4 Quarkonium Spectra

$$\boxed{l = 0}$$

The charge conjugation is give by $c = (-1)^{l+s}$ for the positronium and the photon has $c = -1$

$\boxed{s = 0} \rightarrow \boxed{2\gamma}$

$\boxed{s = 1} \rightarrow \boxed{3\gamma}$

3.4 Quarkonium Spectra

$$\boxed{\left(\Delta E_{quarkonium}\right) \approx 100 MeV}$$

$$\boxed{\left(\Delta E_{positronium}\right) \approx eV}$$

$$\Downarrow$$

Confining potential

$$\boxed{V(r) = -\frac{4}{3}\frac{\alpha_s \hbar c}{r} + F_0 r}$$

3.4 Quarkonium Spectra

$$\boxed{M = m_1 + m_2 + A\frac{\overrightarrow{s_1} \cdot \overrightarrow{s_2}}{m_1 m_2}}$$

Light Mesons

Mesons are bound $q\overline{q}$ states. Here we consider only mesons consisting of LIGHT quarks (u,d,s).

$$m_u \sim 0.3\,\mathrm{GeV},\;\; m_d \sim 0.3\,\mathrm{GeV},$$
$$m_s \sim 0.5\,\mathrm{GeV}$$

Ground state ($L = 0$)

For ground states, where orbital angular momentum is zero, the meson "spin" (total angular momentum) is determined by the $q\overline{q}$ spin state.

Two possible $q\overline{q}$ total spin states $S = (0, 1)$

★ S = 0 : pseudo-scalar mesons

★ S = 1 : vector mesons

Meson Parity : (q and $\overline{q}$ have OPPOSITE parity):

$$\begin{aligned} P &= P(\mathrm{q})P(\overline{\mathrm{q}})(-1)^L \\ &= (+1)(-1)(-1)^L = -1 \quad (\mathrm{for L} = 0) \end{aligned}$$

Flavour States:

$\mathrm{u\bar{d}, u\bar{s}, d\bar{u}, d\bar{s}, s\bar{u}, s\bar{d}}$

$(\mathrm{u\bar{u}, d\bar{d}, s\bar{s}})$ MIXTURES

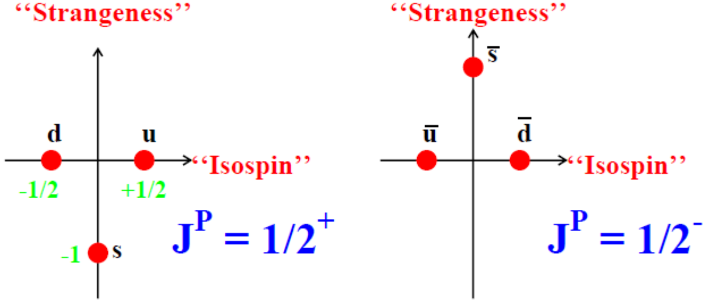
Expect :

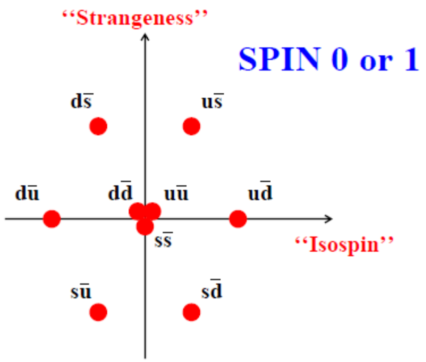
9 $J^P = 0^-$ MESONS : PSEUDO-SCALAR NONET

9 $J^P = 1^-$ MESONS : VECTOR NONET

uds MULTIPLETS

QUARKS ANTIQUARKS





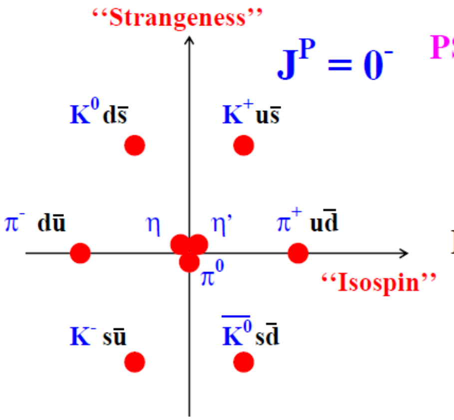
The ideas of strangeness and isospin are historical quantum numbers assigned to different states. Essentially they count quark flavours (this was all before the formulation of the quark model)

Isospin = $\frac{1}{2}(n_{\mathrm{u}} - n_{\mathrm{d}} - n_{\bar{\mathrm{u}}} + n_{\bar{d}})$

Strangeness = $n_{\bar{\mathrm{s}}} - n_{\mathrm{s}}$

Light Mesons

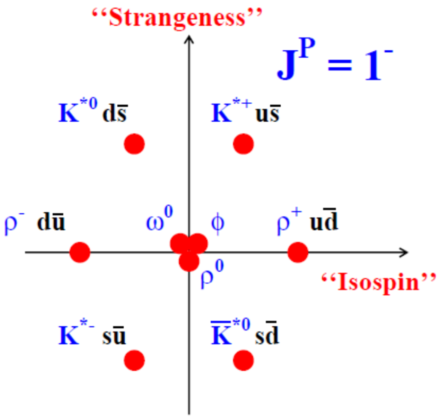
ZERO ORBITAL ANGULAR MOMENTUM



$J^P = 0^-$ **PSEUDOSCALAR NONET**

π^0, η, η' are combinations of $u\bar{u}$, $d\bar{d}$, $s\bar{s}$

Masses/MeV:
$\pi(140)$, $K(495)$
$\eta(550)$, $\eta'(960)$



$J^P = 1^-$ **VECTOR NONET**

ρ^0, ϕ, ω^0 are combinations of $u\bar{u}$, $d\bar{d}$, $s\bar{s}$

Masses/MeV:
$\rho(770)$, $K^*(890)$
$\omega(780)$, $\phi(1020)$

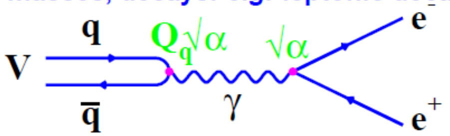
Meson Wave-functions

- ★ $u\bar{d}, u\bar{s}, d\bar{u}, d\bar{s}, s\bar{u}, s\bar{d}$ are straightforward
- ★ However, $(u\bar{u}, d\bar{d}, s\bar{s})$ states all have zero flavour quantum numbers - therefore can MIX

$$\left.\begin{aligned}\pi^0(140) &= \tfrac{1}{\sqrt{2}}(u\bar{u}-d\bar{d})\\ \eta(550) &= \tfrac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s})\\ \eta'(960) &= \tfrac{1}{\sqrt{3}}(u\bar{u}+d\bar{d}+s\bar{s})\end{aligned}\right\} J^P = 0^-$$

$$\left.\begin{aligned}\rho^0(770) &= \tfrac{1}{\sqrt{2}}(u\bar{u}-d\bar{d})\\ \omega^0(780) &= \tfrac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})\\ \phi(1020) &= s\bar{s}\end{aligned}\right\} J^P = 1^-$$

Mixing coefficients determined experimentally from masses, decays. *e.g.* leptonic decays of vector mesons



Matrix element $\propto Q_q\alpha$
Decay rate $\propto Q_q^2\alpha^2$
$\Gamma_{ee} \propto Q_q^2\alpha^2$

$$M_{fi}(\rho^0 \to e^+e^-) \sim e\frac{1}{q^2}\left[\frac{1}{\sqrt{2}}(Q_u e - Q_d e)\right]$$

$$\Gamma_{\rho^0\to e^+e^-} \propto \left[\frac{1}{\sqrt{2}}\left(\frac{2}{3}-\left(-\frac{1}{3}\right)\right)\right]^2 = \frac{1}{2}$$

$$\Gamma_{\omega^0\to e^+e^-} \propto \left[\frac{1}{\sqrt{2}}\left(\frac{2}{3}+\left(-\frac{1}{3}\right)\right)\right]^2 = \frac{1}{18}$$

$$\Gamma_{\phi\to e^+e^-} \propto \left[\frac{1}{3}\right]^2 = \frac{1}{9}$$

PREDICT: $\Gamma_{\rho^0} : \Gamma_{\omega^0} : \Gamma_\phi = 9 : 1 : 2$

EXPERIMENT: $8.8 \pm 2.6 : 1 : 1.7 \pm 0.4$

Meson Masses

Meson masses partly from constituent quark masses

★ $m(K) > m(\pi)$

hints at $m_s > m_u, m_d$

But that is not the whole story

★ $m(\rho^+) > m(\pi^+)$ (770 MeV c.f. 140 MeV)

but both are $u\bar{d}$

★ Only difference is in orientation of Quark SPINS

↑↑ vs. ↓↑

SPIN-SPIN INTERACTION

QED: Hyperfine splitting in H_2 (L=0)

Energy shift due to electron spin in magnetic field of proton

$$\Delta E = \tilde{\mu}.\tilde{\mathrm{B}} = \frac{2}{3}\tilde{\mu}_{\mathrm{e}}.\tilde{\mu}_{\mathrm{p}}|\psi(0)|^2$$

$$\text{using} \quad \tilde{\mu} = \frac{e}{2m}\tilde{\mathrm{S}}$$

$$\Delta E \propto \alpha_{em}\frac{\tilde{\mathrm{S}}_{\mathrm{e}}.\tilde{\mathrm{S}}_{\mathrm{p}}}{m_1 m_2}$$

QCD: Colour Magnetic Interaction

Fundamental form of the interaction between a quark and a gluon is identical to that between an electron and a photon. Consequently, also have a COLOUR MAGNETIC INTERACTION

$$\Delta E \propto \alpha_S \frac{\tilde{\mathrm{S}}_1.\tilde{\mathrm{S}}_2}{m_1 m_2}$$

MESON MASS FORMULA (L=0)

$$\boxed{M_{q\bar{q}} = m_1 + m_2 + A\frac{\tilde{S}_1.\tilde{S}_2}{m_1 m_2}}$$

where A is a constant

For a state of SPIN $\tilde{S} = \tilde{S}_1 + \tilde{S}_2$

$$\begin{aligned}
\tilde{S}^2 &= \tilde{S}_1{}^2 + \tilde{S}_2{}^2 + 2\tilde{S}_1.\tilde{S}_2 \\
\tilde{S}_1.\tilde{S}_2 &= \tfrac{1}{2}(\tilde{S}^2 - \tilde{S}_1{}^2 - \tilde{S}_2{}^2) \\
S_1^2 = S_2^2 &= S_1(S_1+1) = \tfrac{1}{2}(1+\tfrac{1}{2}) = \tfrac{3}{4} \\
\text{giving} \quad \tilde{S}_1.\tilde{S}_2 &= \tfrac{1}{2}\tilde{S}^2 - \tfrac{3}{4}
\end{aligned}$$

For

$J^P = 0^-$ MESONS : $\tilde{S}^2 = 0$

$J^P = 1^-$ MESONS : $\tilde{S}^2 = S(S+1) = 2$

therefore

$$\begin{aligned}
\tilde{S}_1.\tilde{S}_2 &= \tfrac{1}{2}\tilde{S}^2 - \tfrac{3}{4} = -\tfrac{3}{4} \quad (0^- \text{ mesons}) \\
\tilde{S}_1.\tilde{S}_2 &= \tfrac{1}{2}\tilde{S}^2 - \tfrac{3}{4} = +\tfrac{1}{4} \quad (1^- \text{ mesons})
\end{aligned}$$

Giving the (L=0) Meson Mass formulae

$$M = m_1 + m_2 - \frac{3A}{4m_1 m_2} \quad (0^- \text{ mesons})$$

$$M = m_1 + m_2 + \frac{A}{4m_1 m_2} \quad (1^- \text{ mesons})$$

0^- mesons lighter than 1^- mesons

Can now try different values of $m_{u/d}$, m_s and A and try to reproduce the observed values.

	Mass/MeV	
Meson	Predicted	Experiment
π	140	138
K	484	496
ρ	780	770
ω	780	782
K^*	896	894
ϕ	1032	1019

Excellent agreement using:

$m_u = m_d = 310$ MeV,

$m_s = 483$ MeV,

$A = 0.06$ GeV^3.

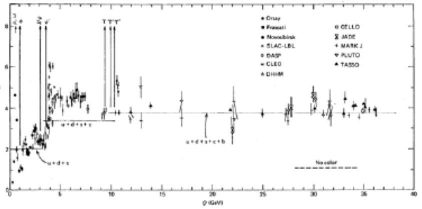
(see Question 4 on the problem sheet)

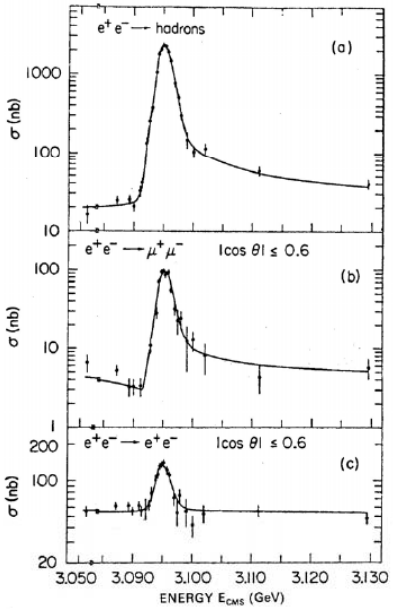
Bring on the Baryons.....

Discovery of the J/ψ

★ 1974 : Discovery of a NARROW RESONANCE in e^+e^- collisions at $\sqrt{s} \approx 3.1$ GeV

Observe resonances R_μ at low $\sqrt{s}$ - many "bumps"





$e^+e^- \to$ hadrons

$\boxed{J/\psi(3097)}$

$e^+e^- \to \mu^+\mu^-$

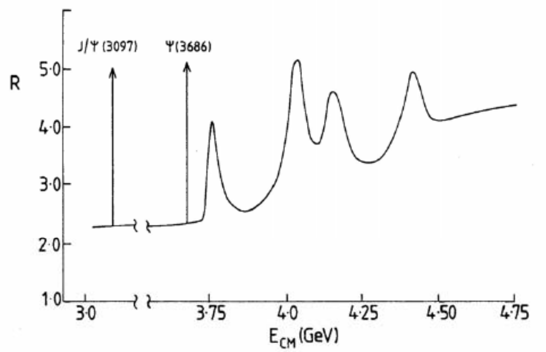
$e^+e^- \to e^+e^-$

Observed width, ~ 3 MeV, all due to experimental resolution ! Actual WIDTH, $\Gamma_{J/\psi} \sim 87\,\text{keV}$.

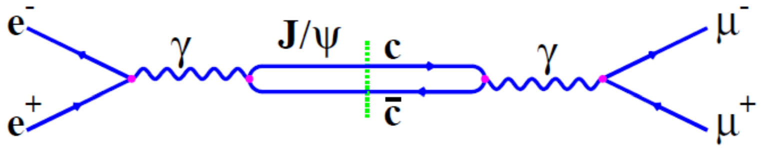
Zoom in to the CHARMONIUM ($c\bar{c}$) region, i.e.

$$\sqrt{s} \sim 2m_c$$

mass of charm quark, $m_c = 1.5\,\mathrm{GeV}$



Resonances due to formation of BOUND unstable $c\bar{c}$ states. The lowest energy of these is the narrow J/ψ state.



The particle physics of decaying particles:

★ Particle Lifetimes
★ Decay Widths
★ Partial Widths
★ Resonances

i.e. the physics of $\mathrm{e^+e^-} \rightarrow \mathrm{Z^0}$.

Much of this will be familiar from Nuclear Physics

3.5 Baryon Wavefunctions

Baryon

1. Three quarks
2. The Pauli's exclusion principle must hold

First, two different tensor products to construct a multiquark wavefunction.

Inner product	Outer product
Not change the number of particles Enlarge the quantum space	Increase the number of particles Not change the quantum space

BARYON WAVE-FUNCTIONS

Baryons made from 3 indistinguishable quarks (flavour treated as another quantum number in the wave-function)

$$\psi_{\text{baryon}} = \psi_{\text{space}}\psi_{\text{flavour}}\psi_{\text{spin}}\psi_{\text{colour}}$$

ψ_{baryon} must be ANTI-SYMMETRIC under interchange of any 2 quarks.

Ground State (L=0)

Here we will only consider the baryon ground states. The ground states have zero ORBITAL ANGULAR momentum,

$$\Rightarrow \quad \psi_{\text{space}} \quad \text{is symmetric}$$

★ All hadrons are COLOUR SINGLETS

$$\psi_{colour} = \frac{1}{\sqrt{6}}(rgb+gbr+brg-grb-rbg-bgr)$$

$$i.e. \quad \psi_{\text{colour}} \quad \text{is anti-symmetric}$$

Therefore $\quad \psi_{\text{space}}\psi_{\text{colour}}$ is anti-symmetric

$\Rightarrow \quad \psi_{\text{spin}}\psi_{\text{flavour}}$ must be SYMMETRIC

★ **Start with the combination of three spin-half particles.**

Trivial to write down the spin wave-function for the $|\frac{3}{2},\frac{3}{2}\rangle$ state :

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle = \uparrow\uparrow\uparrow$$

Generate the other states using $\hat{J}_-$

$$\begin{aligned}
\hat{J}_-|\tfrac{3}{2},\tfrac{3}{2}\rangle &= (\hat{J}_-\uparrow)\uparrow\uparrow + \uparrow(\hat{J}_-\uparrow)\uparrow + \uparrow\uparrow(\hat{J}_-\uparrow) \\
\sqrt{\tfrac{3}{2}\tfrac{5}{2}-\tfrac{3}{2}\tfrac{1}{2}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle &= \downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow \\
|\tfrac{3}{2},\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow)
\end{aligned}$$

Giving the spin $\frac{3}{2}$ states :

$$\begin{aligned}
|\tfrac{3}{2},\tfrac{3}{2}\rangle &= \uparrow\uparrow\uparrow \\
|\tfrac{3}{2},\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow) \\
|\tfrac{3}{2},\text{-}\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow) \\
|\tfrac{3}{2},\text{-}\tfrac{3}{2}\rangle &= \downarrow\downarrow\downarrow
\end{aligned}$$

ALL SYMMETRIC under interchange of any two spins

★ **For the spin-half states, first consider the case where the first two quarks are in a $|0,0\rangle$ state.**

$$\begin{aligned}
|0,0\rangle_{(12)} &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \\
|\tfrac{1}{2},\tfrac{1}{2}\rangle_{(123)} &= |0,0\rangle_{(12)}|\tfrac{1}{2},\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\
|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle_{(123)} &= |0,0\rangle_{(12)}|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)
\end{aligned}$$

ANTI-SYMMETRIC under interchange of $1 \leftrightarrow 2$.

3-quark spin-half states can ALSO be formed from the state with the first two quarks in a SYMMETRIC spin wave-function.

Can construct a 3-particle $|\frac{1}{2},\frac{1}{2}\rangle_{(123)}$ state from

$$|1,0\rangle_{(12)}|\tfrac{1}{2},\tfrac{1}{2}\rangle_{(3)} \quad \text{and}$$
$$|1,1\rangle_{(12)}|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle_{(3)}$$

Taking a linear combination:

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = a|1,1\rangle|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle + b|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

with $a^2+b^2=1$. Act upon both sides with $\hat{\mathrm{J}}_+$:

$$\begin{aligned}
\hat{\mathrm{J}}_+|\tfrac{1}{2},\tfrac{1}{2}\rangle &= a(\hat{\mathrm{J}}_+|1,1\rangle)|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle + a|1,1\rangle(\hat{\mathrm{J}}_+|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle) \\
&\quad + b(\hat{\mathrm{J}}_+|1,0\rangle)|\tfrac{1}{2},\tfrac{1}{2}\rangle + b|1,0\rangle(\hat{\mathrm{J}}_+|\tfrac{1}{2},\tfrac{1}{2}\rangle) \\
0 &= a|1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{2}b|1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle \\
a &= -\sqrt{2}b
\end{aligned}$$

which with $a^2+b^2=1$ implies:

$$a=\sqrt{\tfrac{2}{3}}, \quad b=-\sqrt{\tfrac{1}{3}}$$

Giving

$$\begin{aligned}
|\tfrac{1}{2},\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}|1,1\rangle|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle \\
|\tfrac{1}{2},\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)
\end{aligned}$$

similarly

$$|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow)$$

3 QUARK SPIN WAVE-FUNCTIONS

❶ $\frac{3}{2}$

$$
\begin{aligned}
|\tfrac{3}{2},\tfrac{3}{2}\rangle &= \uparrow\uparrow\uparrow \\
|\tfrac{3}{2},\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow) \\
|\tfrac{3}{2},\text{-}\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow) \\
|\tfrac{3}{2},\text{-}\tfrac{3}{2}\rangle &= \downarrow\downarrow\downarrow
\end{aligned}
$$

SYMMETRIC under interchange of any two quarks

❷ $\frac{1}{2}$

$$
\begin{aligned}
|\tfrac{1}{2},\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\
|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow)
\end{aligned}
$$

SYMMETRIC under interchange of $1 \leftrightarrow 2$

❸ $\frac{1}{2}$

$$
\begin{aligned}
|\tfrac{1}{2},\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow \\
|\tfrac{1}{2},\text{-}\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\downarrow
\end{aligned}
$$

ANTI-SYMMETRIC under interchange of $1 \leftrightarrow 2$

$\psi_{\text{spin}}\psi_{\text{flavour}}$ must be symmetric under interchange of any two quarks.

Consider 3 cases:

❶ Quarks all SAME Flavour: uuu,ddd,sss

★ $\psi_{\rm flavour}$ is SYMMETRIC under interchange of any two quarks.

★ REQUIRE $\psi_{\rm spin}$ to be SYMMETRIC under interchange of any two quarks.

★ ONLY satisfied by SPIN-$\frac{3}{2}$ states.

★ no uuu, ddd, sss, SPIN-$\frac{1}{2}$ baryons with $L = 0$

3 SPIN-$\frac{3}{2}$ states : uuu,ddd,sss.

❷ Two quarks have same Flavour: ddu,uud,..

★ For the like quarks, $\psi_{\rm flavour}$ is SYMMETRIC.

★ REQUIRE $\psi_{\rm spin}$ to be SYMMETRIC under interchange of LIKE quarks $1 \leftrightarrow 2$.

★ satisfied by SPIN-$\frac{3}{2}$ and SPIN-$\frac{1}{2}$

6 SPIN-$\frac{3}{2}$ and **6** SPIN-$\frac{1}{2}$ states: uud, uus, ddu, dds, ssu, ssd

For example: PROTON wave-function $(s_z = \frac{1}{2})$:

$$p\uparrow = \frac{1}{\sqrt{6}}(2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow)$$

The above form for the proton wavefunction is sufficient to derive masses and magnetic moments, however, note that the fully symmetrized wavefunction includes cyclic permutations:

$$\begin{aligned}\frac{1}{\sqrt{18}}(&2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + \\ &2d\downarrow u\uparrow u\uparrow - d\uparrow u\uparrow u\downarrow - d\uparrow u\downarrow u\uparrow + \\ &2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow)\end{aligned}$$

❸ All Quarks have DIFFERENT Flavour: uds

Two possibilities for (ud) part:

i) FLAVOUR SYMMETRIC $\frac{1}{\sqrt{2}}(ud + du)$

★ require spin wave-function to be SYMMETRIC under interchange of ud

★ satisfied by SPIN-$\frac{3}{2}$ and SPIN-$\frac{1}{2}$ states

$\longrightarrow$ ONE SPIN-$\frac{3}{2}$ uds state

$\longrightarrow$ ONE SPIN-$\frac{1}{2}$ uds state

ii) FLAVOUR ANTI-SYMMETRIC $\frac{1}{\sqrt{2}}(ud - du)$

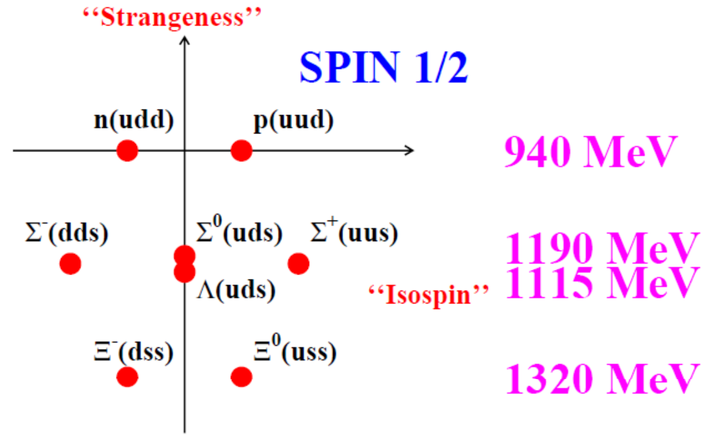
★ require spin wave-function to be ANTI-SYMMETRIC under interchange of ud

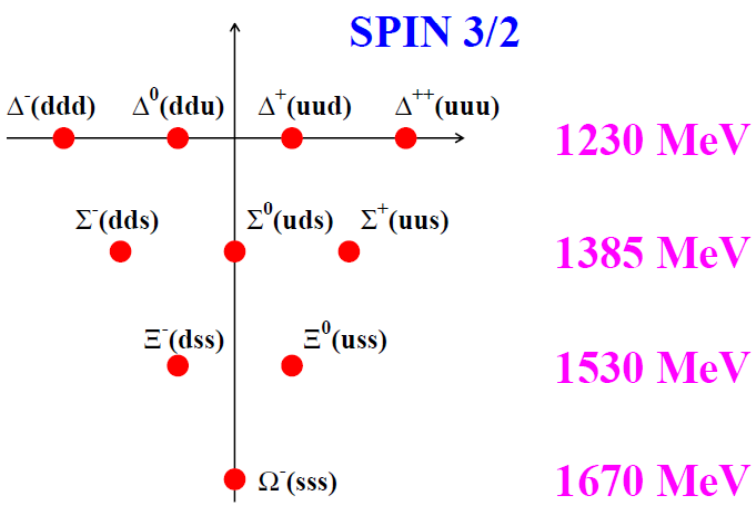
★ only satisfied by SPIN-$\frac{1}{2}$ state

$\longrightarrow$ ONE SPIN-$\frac{1}{2}$ uds state

In total TEN (3+6+1) SPIN-$\frac{3}{2}$ states with the required overall symmetry and EIGHT (0+6+2) SPIN-$\frac{1}{2}$ states

Quark Model predicts Baryons appear in DECUPLETS of SPIN-$\frac{3}{2}$ and OCTETS of SPIN-$\frac{1}{2}$


"Strangeness"
SPIN 1/2
n(udd)
p(uud)
940 MeV
Σ^-(dds)
Σ^0(uds)
Σ^+(uus)
1190 MeV
Λ(uds)
"Isospin"
1115 MeV
Ξ^-(dss)
Ξ^0(uss)
1320 MeV


SPIN 3/2
Δ^-(ddd)
Δ^0(ddu)
Δ^+(uud)
Δ^{++}(uuu)
1230 MeV
Σ^-(dds)
Σ^0(uds)
Σ^+(uus)
1385 MeV
Ξ^-(dss)
Ξ^0(uss)
1530 MeV
Ω^-(sss)
1670 MeV


HADRONS
MESON
BARYON

BARYON MASS FORMULA (L=0)

$$\boxed{M_{\rm qqq} = m_1 + m_2 + m_3 + A'\left(\frac{\tilde{S}_1.\tilde{S}_2}{m_1 m_2} + \frac{\tilde{S}_1.\tilde{S}_3}{m_1 m_3} + \frac{\tilde{S}_2.\tilde{S}_3}{m_2 m_3}\right)}$$

where A' is a constant

EXAMPLE $m_1 = m_2 = m_3 = m_q$

here $$M_{\rm qqq} = 3m_q + A' \sum_{i<j} \frac{\tilde{S}_i.\tilde{S}_j}{m_q^2}$$

$$\begin{aligned}
\tilde{S}^2 &= (\tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3)^2 \\
\tilde{S}^2 &= \tilde{S}_1{}^2 + \tilde{S}_2{}^2 + \tilde{S}_3{}^2 + 2\sum_{i<j} \tilde{S}_i.\tilde{S}_j
\end{aligned}$$

$$\begin{aligned}
2\sum_{i<j} \tilde{S}_i.\tilde{S}_j &= S(S+1) - 3\tfrac{1}{2}(\tfrac{1}{2}+1) \\
2\sum_{i<j} \tilde{S}_i.\tilde{S}_j &= S(S+1) - \tfrac{9}{4} \\
\sum_{i<j} \tilde{S}_i.\tilde{S}_j &= -\frac{3}{4} \qquad J = \tfrac{1}{2} \\
\sum_{i<j} \tilde{S}_i.\tilde{S}_j &= +\frac{3}{4} \qquad J = \tfrac{3}{2}
\end{aligned}$$

e.g. proton(uud) versus Δ(uud)

$$\begin{aligned}
m_p &= 3m_u - \frac{3}{4}\frac{A'}{m_u^2} \\
m_\Delta &= 3m_u + \frac{3}{4}\frac{A'}{m_u^2}
\end{aligned}$$

Again try different values of $m_{u/d}$, m_s and A' and try to reproduce the observed values.

	Mass/MeV	
Baryon	Predicted	Experiment
p/n	939	939
Λ	1116	1114
Σ	1193	1179
Ξ	1318	1327
Δ	1232	1239
Σ^*	1384	1381
Ξ^*	1533	1529
Ω	1672	1682

Excellent agreement using: $m_u = m_d = 363$ MeV,

$m_s = 538$ MeV,

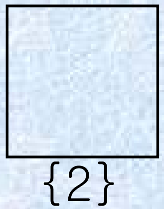
$A' = 0.026$ GeV^3.

QCD predicts $A' = A/2$ where A is the corresponding constant in the meson mass formula.

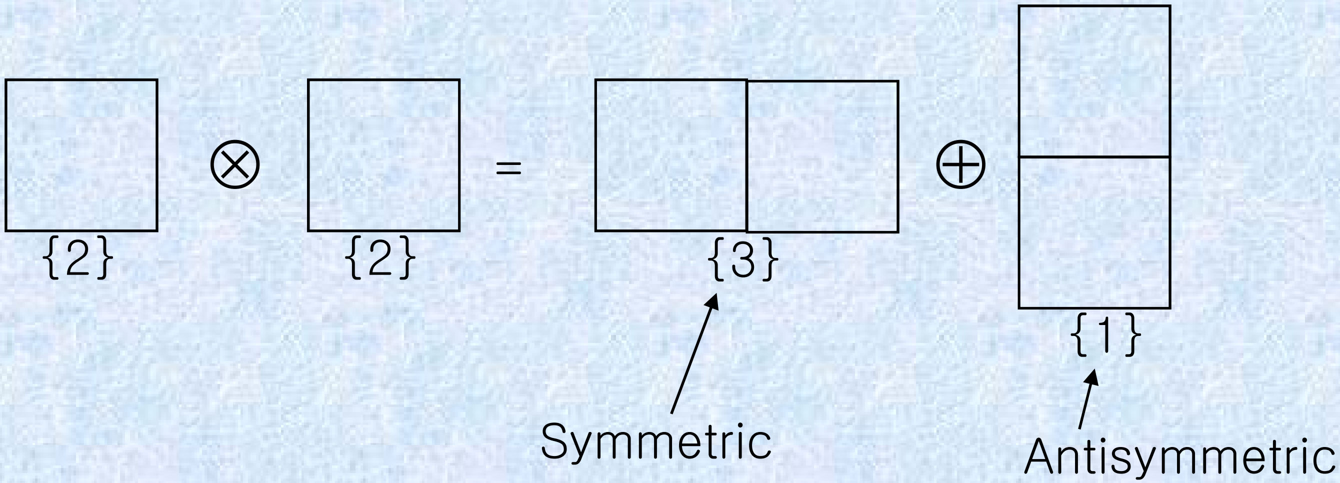
Recall $A = 0.06$ GeV^3 provided a good description of meson masses.

3.5 Baryon Wavefunctions

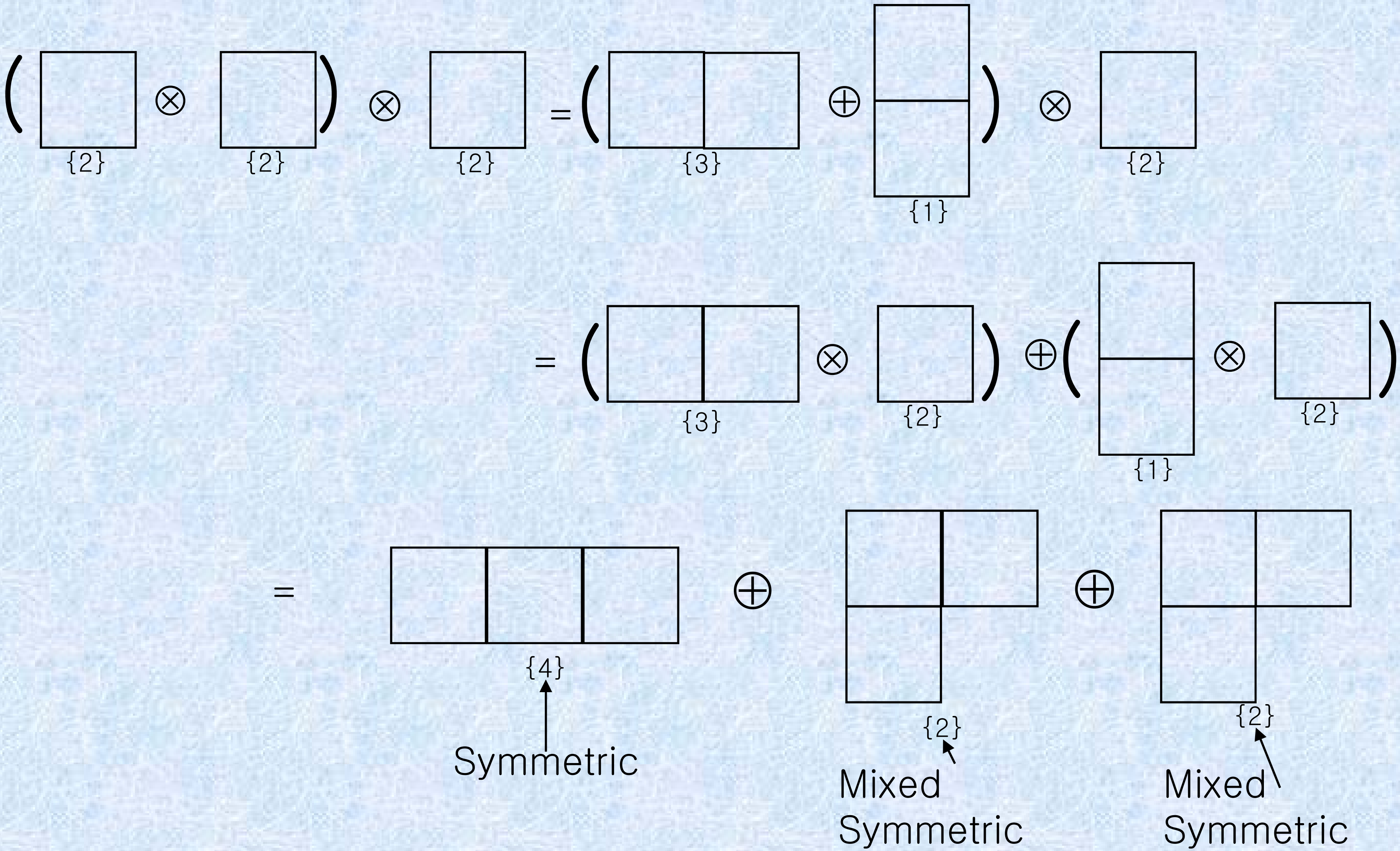
Using the technique of Young tableau.



; doublet meaning ↑ or ↓



3.5 Baryon Wavefunctions

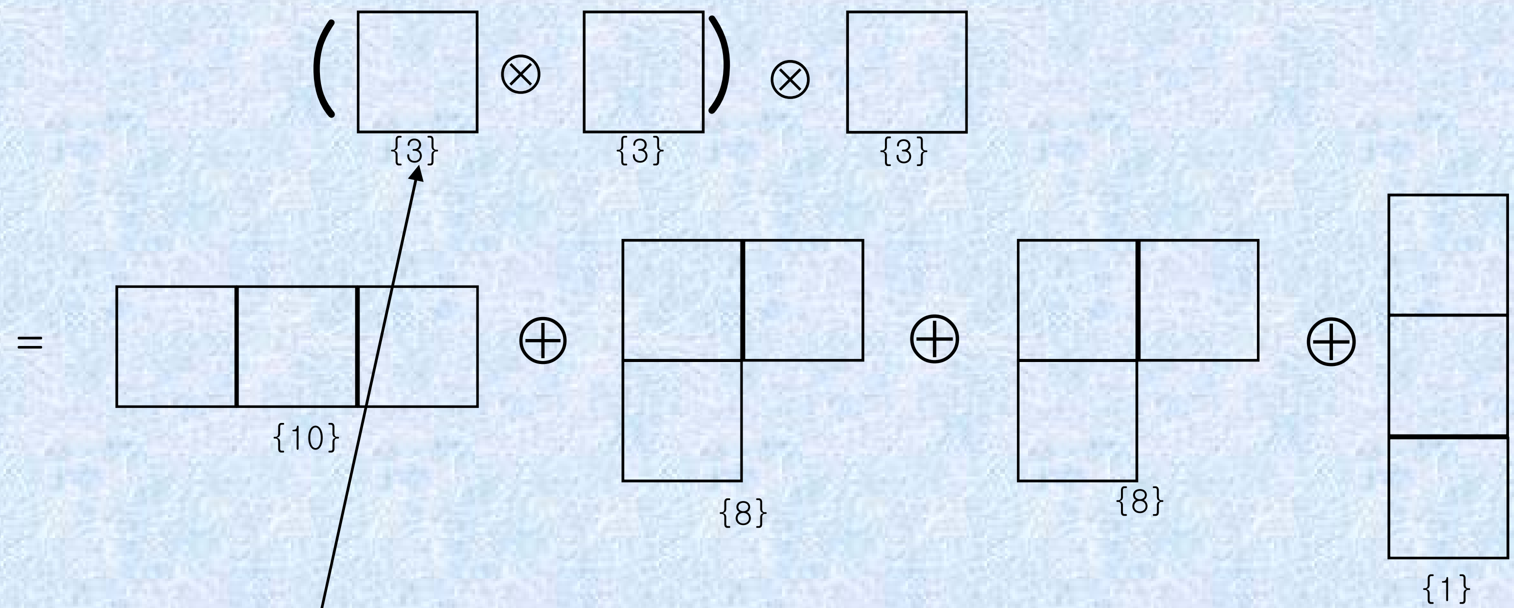


3.5 Baryon Wavefunctions

$$P^{\uparrow} = \begin{array}{|c|c|}\hline \uparrow & \uparrow \\ \hline \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \dfrac{1}{\sqrt{6}}\left(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right) \\ \\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \dfrac{1}{\sqrt{2}}\left(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right) \end{array} \right.$$

<u>Spin wave function!!!</u>

3.5 Baryon Wavefunctions



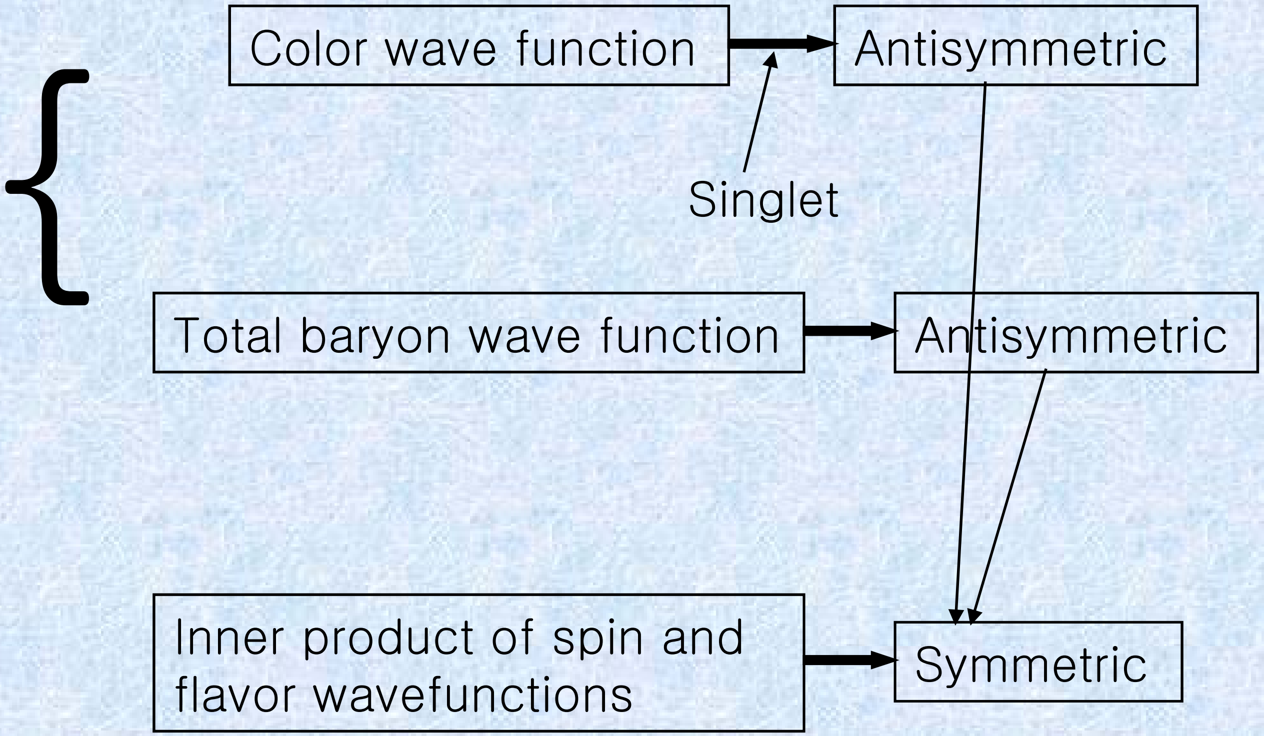
Three flavors, u,d,s

The nucleon (p, n) belong to the octet ({8})

3.5 Baryon Wavefunctions

$$\begin{array}{|c|c|}\hline u & u \\ \hline \multicolumn{1}{|c|}{d} \\ \cline{1-1}\end{array} = \begin{cases} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array} = \dfrac{1}{\sqrt{6}}\left(2uud - udu - duu\right) \\[2em] \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1}\end{array} = \dfrac{1}{\sqrt{2}}\left(udu - duu\right) \end{cases}$$

3.5 Baryon Wavefunctions



3.5 Baryon Wavefunctions

-> Symmetry

Antisymmtry

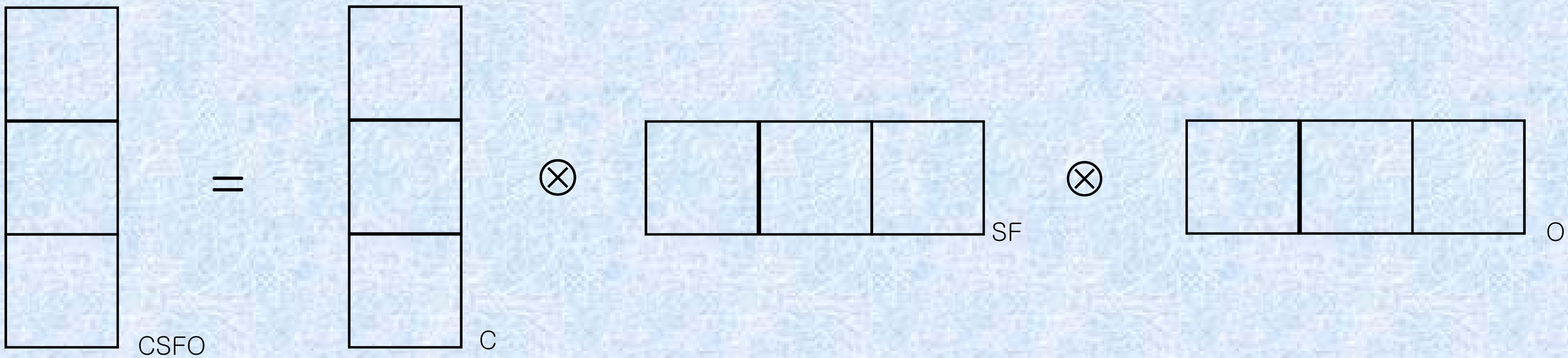
$$\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array}_{SF} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}_{S} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}_{F} \oplus \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}_{S} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}_{F} \right)$$

$$\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array}_{SF} = \frac{1}{\sqrt{2}}\left(\frac{1}{6}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(2uud - udu - duu) + \frac{1}{2}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(udu - duu)\right)$$

$$= \frac{1}{\sqrt{18}}\Big(2(u^{\uparrow}u^{\uparrow}d^{\downarrow} + u^{\uparrow}d^{\downarrow}u^{\uparrow} + d^{\downarrow}u^{\uparrow}u^{\uparrow}) - (u^{\uparrow}u^{\downarrow}d^{\uparrow} + u^{\uparrow}d^{\uparrow}u^{\downarrow} + d^{\uparrow}u^{\uparrow}u^{\downarrow} + u^{\downarrow}u^{\uparrow}d^{\uparrow}$$

$$+ u^{\downarrow}d^{\uparrow}u^{\uparrow} + d^{\uparrow}u^{\downarrow}u^{\uparrow})\Big)$$

3.5 Baryon Wavefunctions



$$\Psi_{p^\uparrow}\left(\vec{r}_1,\vec{r}_2,\vec{r}_3,t\right)=\frac{1}{6\sqrt{3}}\varepsilon_{ijk}\Big(2(u_i^\uparrow u_j^\uparrow d_k^\downarrow+u_i^\uparrow d_j^\downarrow u_k^\uparrow+d_i^\downarrow u_j^\uparrow u_k^\uparrow)$$

$$-(u_i^\uparrow u_j^\downarrow d_k^\uparrow+u_i^\uparrow d_j^\uparrow u_k^\downarrow+d_i^\uparrow u_j^\uparrow u_k^\downarrow+u_i^\downarrow u_j^\uparrow d_k^\uparrow+u_i^\downarrow d_j^\uparrow u_k^\uparrow+d_i^\uparrow u_j^\downarrow u_k^\uparrow)\Big)$$

$$\times\psi_0\left(\vec{r}_1,\vec{r}_2,\vec{r}_3,t\right)$$

S-wave orbital wavefunction

I,j,k are the color indices

Baryon Magnetic Moments

Assume the bound quarks within baryons behave as DIRAC point-like SPIN-1/2 particles with fractional charge, q_q.
Then quarks will have magnetic dipole moments:

$$\hat{\mu}_q = \frac{q_q}{m_q}\hat{S}$$

where m_q is the quark mass.

Magnitude of the magnetic dipole moment:

$$\begin{aligned} \mu_q &= \langle q\uparrow|\frac{q_q}{m_q}\hat{S}|q\uparrow\rangle \\ \text{with}\quad \hat{S}|q\uparrow\rangle &= \frac{1}{2}\hbar|q\uparrow\rangle \\ \text{giving}\quad \mu_q &= \frac{q_q\hbar}{2m_q} \end{aligned}$$

$$\mu_u = \frac{2}{3}\frac{e\hbar}{2m_u}, \mu_d = -\frac{1}{3}\frac{e\hbar}{2m_d}, \mu_s = -\frac{1}{3}\frac{e\hbar}{2m_s}$$

For quarks bound within a L=0 baryon, X, the baryon magnetic moment is the expectation value of the sum of the individual quark magnetic moments.

$$\begin{aligned} \hat{\mu}_B &= \frac{q_1}{m_1}\hat{S}_1 + \frac{q_2}{m_2}\hat{S}_2 + \frac{q_3}{m_3}\hat{S}_2 \\ \mu_X &= \langle X\uparrow|\hat{\mu}_B|X\uparrow\rangle \end{aligned}$$

where $|X\uparrow\rangle$ is the baryon wave-function for the spin up state.

For a spin-up proton:

$$p\uparrow \;=\; \frac{1}{\sqrt{6}}(2u\uparrow u\uparrow d\downarrow - (u\uparrow u\downarrow + u\downarrow u\uparrow)d\uparrow)$$

$$\Rightarrow \mu_p \;=\; \frac{1}{6}\langle(2\uparrow\uparrow\downarrow - (\uparrow\downarrow + \downarrow\uparrow)\uparrow)|\hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3|(2\uparrow\uparrow\downarrow - (\uparrow\downarrow + \downarrow\uparrow)\uparrow)\rangle$$

Consider the contribution from quark 1 (an up-quark):

$$\begin{aligned}
& \frac{1}{6}\langle(2\uparrow\uparrow\downarrow - (\uparrow\downarrow + \downarrow\uparrow)\uparrow)|\hat{\mu}_1|(2\uparrow\uparrow\downarrow - (\uparrow\downarrow + \downarrow\uparrow)\uparrow)\rangle \\
= & \frac{1}{6}\langle(2\uparrow\uparrow\downarrow - (\uparrow\downarrow + \downarrow\uparrow)\uparrow)|(2\mu_1\uparrow\uparrow\downarrow - (\mu_1\uparrow\downarrow - \mu_1\downarrow\uparrow)\uparrow)\rangle \\
= & \frac{2}{3}\mu_1 = \frac{2}{3}\mu_u = \frac{4}{9}\frac{e\hbar}{2m_u}
\end{aligned}$$

Summing over the other contributions gives:

$$\begin{aligned}
\mu_p = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d \;&=\; \frac{4}{9}\frac{e\hbar}{2m_u} + \frac{4}{9}\frac{e\hbar}{2m_u} + \frac{1}{9}\frac{e\hbar}{2m_d} \\
&=\; \frac{e\hbar}{2m_{u/d}} = \frac{m_p}{m_{u/d}}\mu_N
\end{aligned}$$

where μ_N is the NUCLEAR MAGNETON $\mu_N = e\hbar/2m_p$

Repeat for other (L=0) Baryons, PREDICT

$$\boxed{\frac{\mu_n}{\mu_p} = -\frac{2}{3}}$$

compared to the experimentally measured value of

$$-0.685$$

Baryon	μ_B in Quark Model	Predicted $[\mu_N]$	Experiment $[\mu_N]$
p	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_d$	$+2.79$	$+2.793$
n	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_u$	-1.86	-1.913
Λ	μ_s	-0.61	-0.614 ± 0.005
Σ^+	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	$+2.68$	$+2.46 \pm 0.01$
Ξ^0	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_u$	-1.44	-1.25 ± 0.014
Ξ^-	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_d$	-0.51	-0.65 ± 0.01
Ω^-	$3\mu_s$	-1.84	-2.02 ± 0.05

Impressive agreement with data using
$m_u = m_d = 336$ MeV and
$m_s = 509$ MeV.

(see Question 5 on the problem sheet)

SUMMARY

★ Baryons and mesons are complicated objects.

★ However, the Quark model can be used to make predictions for masses/magnetic moments.

★ The predictions give reasonably consistent values for the constant quark masses.

	$m_{u/d}$	m_s
Meson Masses	310 MeV	483 MeV
Baryon Masses	363 MeV	538 MeV
Baryon mag. moms.	336 MeV	510 MeV

$$\boxed{m_u \approx 335 \text{ MeV}}$$

$$\boxed{m_d \approx 335 \text{ MeV}}$$

$$\boxed{m_s \approx 510 \text{ MeV}}$$

★ What about the HEAVY quarks (charm, bottom, top)....

References

- Seok Hoon Yun
- M.A. Thomson
- …

Thank you.