Mathematical Physics

Jungil Lee
Department of Physics, Korea University
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Textbook:

Arfken and Weber, Essitial Mathematical Methods for Physicists.

Chaper 4
Group Theory

Definition of Group G

- closure under multiplication : if $a, b \in G$, then $ab \in G$.
- multiplication is associative : $(ab)c = a(bc)$.
- $\exists 1 \in G$ such that $1a = a1 = a \ \forall a \in G$.
- $\exists a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = 1 \ \forall a \in G$.

Example)

- Show that $\{1\}$ is a group.
- Show that $\{1, i, -i, -1\}$ is a group.

- Show that 1 is unique : Assume $\exists 1' \in G$ and $1' \neq 1$ such that $1'a = a1' = a \ \forall a \in G$. Then you will find the assumption is wrong.
  $(1'1 = 11' = 1) \wedge (1'1 = 11' = 1') \rightarrow (1 = 1')$
- Show that a^{-1} is unique : Assume $\exists (a^{-1})' \in G$ and $(a^{-1})' \neq a^{-1}$ such that $(a^{-1})'a = a(a^{-1})' = 1 \ \forall a \in G$. Then you will find the assumption is wrong.
  $a(a^{-1})' = 1 \rightarrow a^{-1}[a(a^{-1})'] = a^{-1}1 \rightarrow [a^{-1}a](a^{-1})' = a^{-1} \rightarrow (a^{-1})' = a^{-1}$.

Example 1 2-Dimensional rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\phi) \begin{pmatrix} x \\ y \end{pmatrix}, \quad R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

- Show that $R(\phi)$ is orthogonal[each row(cloumn) is orthogonal to the others].
- Show that $\mathrm{Det}[R(\phi)] = +1$.
- Show that $[R(\phi)]^{-1} = [R(\phi)]^T = R(-\phi)$.
- Show that $G = \{R(\phi)\}$ is a group.

- Show that $G = \{R(\phi)\}$ is abelian(commutative);
  $R(\phi_1)R(\phi_2) = R(\phi_2)R(\phi_1)$.
- Show that G is SO(2); special orthogonal group (2×2).
- Subgroup is a group inside a group.
- Show that $\{R(0), R(\pi)\}$ is a subgroup in G.
- Show that $\{R(0), R(\frac{\pi}{2}), R(\pi), R(\frac{3\pi}{2})\}$ is a subgroup in G.

invariant subgroup

- G' is an invariant subgroup of G if $gg'g^{-1} \in G'$ $\forall g \in G$ and $\forall g' \in G'$.
- Show that $\{R(0), R(\pi)\}$ is an invariant subgroup in G.
- Show that $\{R(0), R(\frac{\pi}{2}), R(\pi), R(\frac{3\pi}{2})\}$ is an invariant subgroup in G.

Example 4.1.2) Similarity transformation $\{R_x(\phi)\}$, $\{R_y(\phi)\}$, and $\{R_z(\phi)\}$ are subgroups or order 2 in SO(3).

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}, \quad R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix},$$

$$R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

- Show that $R_x(\frac{\pi}{2})R_z(\phi)[R_x(\frac{\pi}{2})]^{-1} = R_y(\phi)$.
- Therefore $\{R_z(\phi)\}$ is not an invariant subgroup.

special orthogonal group SO($\boldsymbol{n}$)

- Show that $(AB)^T = B^T A^T$ for any matrices A and B.
- Show that $(AB)^{-1} = B^{-1} A^{-1}$ for any matrices A and B.
- Show that $O_i^{-1} = O_i^T$ if $\{O_i\}$ is a SO(n) group.
- Show that $(O_1 O_2)^{-1} = (O_1 O_2)^T$; if O_1 and O_2 are orthogonal, then $O_1 O_2$ is also an orthogonal matrix.
- Show that real orthogonal $n \times n$ matrix has $\frac{1}{2}n(n-1)$ independent parameters.
- Show that SO(2) has only one independent parameter.
- Show that SO(3) has three independent parameters such as Euler angles.

Number of independent parameters of a group

- General linear group made of real $n \times n$ matrix, $\boldsymbol{GL(n, R)}$, has n^2 real elements. Show that There are $\boldsymbol{n^2}$ independent real parameters.
- General linear group made of complex $n \times n$ matrix, $\boldsymbol{GL(n, C)}$, has n^2 complex elements. Show that there are $\boldsymbol{2n^2}$ independent real parameters.
- Special linear group made of real $n \times n$ matrix with determinant$= +1$, $\boldsymbol{SL(n, R)}$, has n^2 real elements and, therefore, the determinant must be real. Show that the condition "determinant$= +1$" eliminates one parameter. There are $\boldsymbol{n^2 - 1}$ independent real parameters.

- Special linear group made of complex $n \times n$ matrix with determinant$= +1$, $\boldsymbol{SL(n, C)}$, has n^2 complex elements and, therefore, the determinant is in general complex. Show that the condition "determinant$= +1$" eliminates two real parameters. There are $\mathbf{2(n^2 - 1)}$ independent real parameters.
- Show that $GL(n, C) \supset GL(n, R) \supset SL(n, R)$.
- Show that $GL(n, C) \supset SL(n, C) \supset SL(n, R)$.
- Unitary group made of complex $n \times n$ matrix $UU^\dagger = U^\dagger U = 1$, $\boldsymbol{SL(n, C)}$, has n^2 complex elements u_{ij}. Show that the diagonal terms of $UU^\dagger = U^\dagger U$ are always real and equal to 1; n constraints. Show that the off-diagonal terms of $UU^\dagger = U^\dagger U$ are in general complex and equal to 0; $\frac{1}{2}n(n-1) \times 2$ constraints.

- Show that $UU^\dagger = U^\dagger U = 1$ generates n^2 constraints and, therefore, we have $\boldsymbol{n^2}$ independent parameters for $U(n)$.

- Show that Det(AB) =Det$(A)\cdot$Det(B) for any matrices A and B.

- Show that Det(A^T) =Det(A) for any matrix A.

- Show that Det$(A^{-1}) = 1/$Det(A) for any matrix A.

- Show that Det$(A^\dagger)$ =[Det(A)]*.

- Show that if $UU^\dagger = U^\dagger U = 1$, then Det$(U)\cdot$[Det$(U)$]$^* = \left|\text{Det}(U)\right|^2 = 1 \to$ Det$(U) = e^{i\theta}$ and θ is a free real parameter.

- Show that special unitary group $\boldsymbol{SU(n)}$ of $n \times n$ complex matrix with the conditions $UU^\dagger = U^\dagger U$ and Det$(U) = +1$.

Show that the constraint $\mathrm{Det}(U) = +1$ kills the parameter θ for $\mathrm{Det}(U) = e^{i\theta}$ thus $\boldsymbol{SU(n)}$ has $\boldsymbol{n^2 - 1}$ free parameters.

- Complex orthogonal group $O(n, C)$ is made of $n \times n$ complex matrix O with $OO^T = O^T O = 1$.

- Show that $\mathrm{Det}(OO^T) = \mathrm{Det}(1)$ leads to $\mathrm{Det}(O) = \pm 1$. The determinant of an orthogonal matrix is determined and it is not a free parameter. We may choose the sign $+1$ or -1. The two matrices are completely independent. Show that the two matrices are NOT related by any similarity transformation $U_- = PU_+P^{-1}$, where $U_\pm$ is a unitary matrix with determinant ± 1. You can check it by taking determinant of both sides.

- Show that diagonal components of $OO^T = O^T O = 1$ gives n complex equations, $\sum_{k=1}^n o_{ik}^2 = 1 + i0$, where $i = 1, \cdots, n$. The condition eliminates $2n$ free parameters.

- Show that off-diagonal components of $OO^T = O^T O = 1$ gives $\frac{1}{2}n(n-1)$ complex equations, $\sum_{k=1}^{n} o_{ik}o_{jk} = 0 + i0$, where $i, j = 1, \cdots, n$. The condition eliminates $n(n-1)$ free parameters.
- Show that the number of constrants for the complex orthogonal group is $n(n+1)$. Therefore $\boldsymbol{O(n, C)}$ has $\boldsymbol{n(n-1)}$ free real parameters.
- Real orthogonal group $O(n, R)$ is made of $n \times n$ real matrix O with $OO^T = O^T O = 1$.
- Show that $\mathrm{Det}(OO^T) = \mathrm{Det}(1)$ leads to $\mathrm{Det}(O) = \pm 1$. The determinant of an orthogonal matrix is determined and it is not a free parameter. We may choose the sign $+1$ or -1. The

two matrices are completely independent. Show that the two matrices are NOT related by any similarity transformation $U_- = PU_+P^{-1}$, where $U_\pm$ is a unitary matrix with determinant ± 1. You can check it by taking determinant of both sides.

- Show that diagonal components of $OO^T = O^TO = 1$ gives n real equations, $\sum_{k=1}^n o_{ik}^2 = 1$, where $i = 1, \cdots, n$. The condition eliminates n free parameters.

- Show that off-diagonal components of $OO^T = O^TO = 1$ gives $\frac{1}{2}n(n-1)$ real equations, $\sum_{k=1}^n o_{ik}o_{jk} = 0$, where $i, j = 1, \cdots, n$. The condition eliminates $\frac{1}{2}n(n-1)$ free parameters.

- Show that the number of constrants for the real orthogonal group is $\frac{1}{2}n(n+1)$. Therefore $\boldsymbol{O(n,R)}$ has $\boldsymbol{\frac{1}{2}n(n-1)}$ free real parameters.

- Show that special orthogonal group $SO(n,C)$, a group made of the elements of $O(n,C)$ with determinant$=+1$, is a subgroup of $O(n,C)$. Show that $SO(n,C)$ has has $\boldsymbol{n(n-1)}$ free real parameters like $O(n,C)$.

- Elements of $O(n,C)$ with determinant$=-1$ do not make a group. You can check it by taking determinant of a product of two matrices with determinant$=-1$ to find it is 1 instead of -1. Show that there are $\boldsymbol{n(n-1)}$ free real parameters in this space.

- Show that special orthogonal group $SO(n,R)$, a group made of the elements of $O(n, R)$ with determinant$= +1$, is a subgroup of $O(n, C)$. Show that $SO(n, R)$ has has $\boldsymbol{\frac{1}{2}n(n-1)}$ free real parameters like $O(n, R)$.

- Elements of $O(n, R)$ with determinant$= -1$ do not make a group. You can check it by taking determinant of a product of two matrices with determinant$= -1$ to find it is 1 instead of -1. Show that there are $\boldsymbol{\frac{1}{2}n(n-1)}$ free real parameters in this space.

Euler angles

Show that

$$A(\alpha,\beta,\gamma) \equiv R_z(\gamma)R_y(\beta)R_z(\alpha)$$

$$= \begin{pmatrix} +c_\gamma c_\beta c_\alpha - s_\gamma s_\alpha & c_\gamma c_\beta s_\alpha + s_\gamma c_\alpha & -c_\gamma s_\beta \\ -s_\gamma c_\beta c_\alpha - c_\gamma s_\alpha & -s_\gamma c_\beta s_\alpha + c_\gamma c_\alpha & s_\gamma s_\beta \\ s_\beta c_\alpha & s_\beta s_\alpha & c_\beta \end{pmatrix},$$

where $c_\alpha = \cos\alpha$ and $s_\alpha = \sin\alpha$, makes a SO(3) group.

- Find α, β, γ such that $A(\alpha,\beta,\gamma) = R_x(\theta)$.
- Find α, β, γ such that $A(\alpha,\beta,\gamma) = R_y(\theta)$.
- Find α, β, γ such that $A(\alpha,\beta,\gamma) = R_z(\theta)$.

special unitary group SU($\boldsymbol{n}$)

- Determinant is +1: special.
- $U^{-1} = U^{\dagger}$: unitary.
- Complex $n \times n$ unitary matrix has $n^2 - 1$ degrees of freedom; $2n^2 - n^2_{\text{unitarity}} - 1_{\text{Det}=1} = n^2 - 1$.
- Show that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ for any matrices A and B.
- Show that U_1U_2 is unitary if U_1 and U_2 are unitary.

Let us show that complex $n \times n$ unitary matrix (a_{ij}) with positive determinant has $n^2 - 1$ independent parameters.

- Originally we have n^2 complex ($2n^2$ real) parameters because the matrix (a_{ij}) is $n \times n$ and complex.
- Unitarity gives constraints $\sum_k (a^\dagger)_{ik} a^{kj} = \sum_k a^*_{ki} a_{kj} = \delta_{ij}$.
- for $i = j$ we have n conditions $\sum_k (a^\dagger)_{ik} a^{ki} = \sum_k a^*_{ki} a_{ki} = \sum_k |a_{ki}|^2 = 1$. Note that this constraints are equations for real numbers because both side are real numbers; The sum of real numbers $|a_{ki}|^2$ is real.
- for $i \neq j$ we have $n(n-1)$ conditions. Note that there are $\frac{1}{2}n(n-1)$ equations and the left-hand side $\sum_k a^*_{ki} a_{kj}$ is a complex number. Thus we have $n + n(n-1) = n^2$ constraints

from the unitarity condition.

- The determinant is $+1$. This is one more constraint.
- Subtracting the number of constraints from the number of original parametes, we get the number of independent parameters $2n^2 - (n^2 + 1) = n^2 - 1$.

Pauli matrices and special unitary group SU(2)

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Show that $\mathrm{Tr}(\sigma_i) = 0$.
- Show that $\sigma_i \sigma_j = \delta_{ij} 1 + i\epsilon_{ijk}\sigma_k$.
- Show that $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$.
- Show that $\{\sigma_i, \sigma_j\} = 2\delta_{ij}1$.
- Show that $\boldsymbol{a} \cdot \boldsymbol{\sigma}\ \boldsymbol{b} \cdot \boldsymbol{\sigma} = \boldsymbol{a} \cdot \boldsymbol{b}\ 1 + i\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{\sigma}$.
- Show that any Hermitian 2×2 matrix H is expressed as $H = \frac{1}{2}\mathrm{Tr}(H)\ 1 + \frac{1}{2}\mathrm{Tr}(H\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma}$.

Example 4.1.3 Show that $G = \{e^{i\theta}, \theta \in R\}$ is a unitary group U(1); U= unitary, (1) single parameter.

- $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \in G$.
- $(e^{i\theta_1}e^{i\theta_2})e^{i\theta_3} = e^{i\theta_1}(e^{i\theta_2}e^{i\theta_3}) = e^{i(\theta_1+\theta_2+\theta_3)} \in G$.
- $e^{i0} = 1 \in G$.
- $(e^{i\theta_1})^{-1} = e^{-i\theta_1} = (e^{i\theta_1})^{\dagger} \in G$.
- Show that $\{1, -1\}$ is a subgroup of G.
- Show that $\{1, -1, i, -i\}$ is a subgroup of G.
- Show that $\{1, \sigma_1\}$, $\{1, \sigma_2\}$, and $\{1, \sigma_3\}$, are subgroups of $SU(2)$; Use $\sigma_k^2 = 1 \; \forall \; k = 1, 2, 3$.

Homomorphism) Consider two groups G and H. There is a transform $H = \{h = f(g), g \in G\}$. If $f(g_1 g_2) = f(g_1)f(g_2)$, the two groups are homomorphic.

- Show that G and H are homomorphic If
  $H = \{h = UgU^{-1}, g \in G\}$ and $G = \{g\}$;
  $h_1 h_2 = (Ug_1U^{-1})(Ug_2U^{-1}) = U(g_1g_2)U^{-1}$.

Isomomorphism) Consider two groups G and H. If G and H are homomorphic and there is one-to-one correspondence, they are isomorphic.

- Show that G and H are homomorphic If
  $H = \{h = UgU^{-1}, g \in G\}$ and $G = \{g\}$;
  $h_1 h_2 = (Ug_1U^{-1})(Ug_2U^{-1}) = U(g_1g_2)U^{-1}$.

Diagonalization

- Solve the eigenvalue problem $A|x_i\rangle = \lambda_i|x_i\rangle$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, to find $\lambda_1 = +1$, $\lambda_2 = -1$ with $|x_1\rangle = (1, +1)^T$ and $|x_2\rangle = (1, -1)^T$.
- Show that PAP^{-1} =diag(λ_1, λ_2) is diagonal, where $P^{-1} = (|x_1\rangle, |x_2\rangle)$.
- Show that $P|x_1\rangle = (1, 0)^T$ and $P|x_2\rangle = (0, 1)^T$.

Reducible representation If a matrix is block-diagonalizable, it is reducible.

- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that $PAP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is diagonal, where $P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Ireducible representation Fully block-diagonalized matrix representation.

Time-independent Schrödinger equation $H\psi = E\psi$.

- if H is invariant under the similarity transformation $RHR^{-1} = H$, $[H, R] = 0$.
- if $[H, R] = 0$, ψ and $R\psi$ are degenerate; have a common eigenvalue.

Multiplet; basis vectors of a vector space

- spin doublet; spin $\uparrow$ and spin $\downarrow$ states.
- $2\ell + 1$-plet; $|J = \ell, J_z = m_\ell = -\ell\rangle, \cdots, |J = \ell, J_z = m_\ell = \ell\rangle$

Matrix representation: ψ_i, $i = 1, \cdots, n$ are basis vectors of a vector space V_ψ.

$$(R\psi)_j = \sum_k r_{jk}\psi_k, \quad R \in G$$

r_{jk} is the matrix represenation of G with the basis $\{\psi_i \mid i = 1, \cdots, n\}$.

Irreducible representation: if $\{R\psi_i\} = V_\psi$ $\forall \psi_i \in V_\psi$ and $\forall R \in G$, then the representation is irreducible.

Reducible representation: not irreducible.

Direct sum: If V_ψ is reducible and V_i are irreducible, then $\exists$ a unitary transform U such that $UrU^\dagger$ is block-diagonalized as

$$UrU^\dagger = \begin{pmatrix} \boldsymbol{r}_1 & \mathbf{0} & \ldots & \\ \mathbf{0} & \boldsymbol{r}_2 & \mathbf{0} & \\ \vdots & \mathbf{0} & \ddots & \end{pmatrix}$$

And V_ψ is a direct sum of V_i

$$V_\psi = V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1} \oplus V_n$$

- Show that

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$$
$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)$$

- Show that

$$\cos(i\alpha) = \cosh\alpha, \quad \sin(i\alpha) = i\sinh\alpha.$$

- Show that

$$\cosh\alpha\cosh\beta + \sinh\alpha\sin\beta = \cosh(\alpha+\beta)$$
$$\sinh\alpha\cosh\beta + \cosh\alpha\sin\beta = \sinh(\alpha+\beta)$$

- Show that $L(\alpha)L(\beta) = L(\alpha+\beta) = L(\beta)L(\alpha)$ where

$$L(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$

- Show that $\{L(\alpha)\}$ is an abelian group.
- Show that $[L(\alpha)]^{-1} = L(-\alpha)$.

4.2 Generators of Continuous Group

- Show that $(\sigma_k)^n = \delta_{n,\text{even}} 1 + \delta_{n,\text{odd}} \sigma_k$.
- Prove the Euler's identity
  $e^{i\sigma_k \theta} \equiv \sum_{n=0}^{\infty} \frac{(i\sigma_k \theta)^n}{n!} = 1 \ \cos\theta + i\sigma_k \sin\theta$.
- Show that
  $$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = 1 \ \cos\phi + i\sigma_2 \sin\phi = e^{i\sigma_2 \phi}.$$
- Using Euler's identity, show that $e^{i\sigma_k \phi_1} e^{i\sigma_k \phi_2} = e^{i\sigma_k (\phi_1 + \phi_2)}$

exponential function of a matrix

- Show that $\ln(1+x) = -\sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$.
- Show that $\lim_{n\to\infty} n \ln\left(1+\frac{x}{n}\right) = x$.
- Show that $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$.
- Show that $e^{i\phi S} = \lim_{n\to\infty} \left(1+\frac{i\phi}{n}S\right)^n$, where S is a matrix.

Beker-Housdorff formula : Consider $O = e^{i\phi S} A e^{-i\phi S}$.

- Show that $\frac{\partial}{\partial \phi} O = e^{i\phi S}\ i[S, A]\ e^{-i\phi S}$.
- Show that $\frac{\partial^n}{\partial \phi^n} O = e^{i\phi S}\ i^n\ f_n(S, A)\ e^{-i\phi S}$, where $f_{n+1}(S, A) = [S, f_n(iS, A)]$ and $f_0(S, A) = A$.
- Prove the Beker-Housdorff formula $O = \sum_{n=0}^{\infty} f_n(S, A) \frac{(i\phi)^n}{n!}$.
- Using the Beker-Housdorff formula, show that $e^{i\phi S} e^{-i\phi S} = 1$; $\left(e^{i\phi S}\right)^{-1} = e^{-i\phi S}$.
- Using the Beker-Housdorff formula, show that $e^{i\phi S} A e^{-i\phi S} = A$, if A and S commute.

Generators of a group: Consider a group element $R = e^{i\phi S} \in G$, where $\mathrm{Det}(R) = +1$. Assume S is diagonalizable; $USU^{-1} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

- Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ $\forall A$ and B.
- Show that $\mathrm{Tr}(URU^{-1}) = \mathrm{Tr}(R)$.
- Show that $\mathrm{Det}(R) = \mathrm{Det}(URU^{-1}) = \mathrm{Det}(e^{U(i\phi S)U^{-1}}) = \mathrm{Det}[\mathrm{diag}(e^{i\phi\lambda_1}, \cdots, e^{i\phi\lambda_n})] = e^{i\phi \mathrm{Tr} S}$.
- Show that S is traceless; $\mathrm{Tr}(S) = 0$.
- Show that if R is unitary, then S is Hermitian.(ϕ is real)
- Show that if R is real orthogonal, then S is Hermitian and pure imaginary.(ϕ is real)

Consider a group $R = e^{i\sum_k \phi_k S_k} \in G$ of order r, where $\mathrm{Det}(R) = +1$. There are r independent parameters of transformation. We call S_k's generators of the group.

- Show that $\mathrm{Det}(S_i)$ does not have to be $+1$ unlike R.
- Show that the number of generators is always same as the order of a group. Hint: count the number of constraints and compare with that for the group.
- Show that $[S_i, S_j]$ is antihermitian.
- Show that $\{S_i, S_j\} \equiv S_i S_j + S_j S_i$ does not have to be traceless.
- Show that $\forall$ antihermitian A, $B = B^\dagger$, where $A = iB$.
- Show that $[S_i, S_j]$ is traceless.

- Show that if G is SU(n), there are $n^2 - 1$ generators.
- Show that if G is SO(n), there are $n(n-1)/2$ generators.
- Show that any traceless Hermitian matrix can be expressed as a linear combination of $\{S_i\}$.
- Show that $[S_i, S_j]$ can be expressed in a linear combination of S_k's. $[S_i, S_j] = i \sum_k c_{ijk} S_k$, where the real numbers c_{ijk}'s are the structure constants of the group.

- Show that if A and B are Hermitian, $\{A, B\}$ is Hermitian.
- Show that if A is Hermitian, eigenvalues are real. Hint: $H|\psi\rangle = \lambda|\psi\rangle \rightarrow \langle\psi|H|\psi\rangle = \lambda\langle\psi|\psi\rangle$. Take the complex conjugate.
- Show that if A is Hermitian of dimension n, one can choose n eigenvectors, where any two are orthogonal to each other; they make a basis set. Therefore, A is diagonalizable.
- Show that if A is Hermitian, $\mathrm{Tr}(A)$ is real.

- Show that $\mathrm{Tr}(S_i S_j) = \frac{1}{2}\mathrm{Tr}(S_i S_j + S_j S_i) = f_{ij}$ is real and symmetric under exchange of the two indices.
- Show that $\mathrm{Tr}(S_i S_j) = f_{ij}$ is diagonalizable.
- Show that once $\mathrm{Tr}(S_i S_j) = f_{ij}$ is diagonalized, one can choose the normalization so that $\mathrm{Tr}(S'_i S'_j) = \lambda\delta_{ij}$.
- $\mathrm{Tr}[[S_i, S_j], S_k]$ is totally antisymmetric under exchange of any two indices.
- Show that if $\mathrm{Tr}(S_i S_j) = \lambda\delta_{ij}$, c_{ijk} is totally antisymmetric under exchange of any two indices. Hint: $\mathrm{Tr}([[S_i, S_j], S_k]) = 2i\lambda c_{ijk}$.
- Show that the structure constant is independent of representation; c_{ijk} is invarinat under PS_iP^{-1}.

Hamiltonian operator and time evolution

- Show that $f(x+a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{\partial^n}{\partial x^n} f(x) = e^{\pm a \frac{\partial}{\partial x}} f(x)$.
- Show that $H = i\frac{\partial}{\partial t}$ is the generator for the time evolution; $U(\Delta t)\psi(t) = \psi(t+\Delta t)$, where $U(\Delta t) = e^{-iH\Delta t}$.
- Show that if $H\psi(t) = E\psi(t)$, then $\psi(t) = e^{-iE(t-t_0)}\psi(t_0)$.
- Show that $U^{-1}(\Delta t) = U^{\dagger}(\Delta t) = U(-\Delta t)$.
- Show that $U(\Delta t)HU^{\dagger}(\Delta t) = H$.
- Show that for a free particle($H = \frac{p_x^2}{2m}$) moving along the x-axis, $[H, p_x] = 0$ and therefore $U(\Delta t)p_x U^{\dagger}(\Delta t) = p_x$. Thus $e^{i(p_x x - Et)}$ is the eigenstate of both H and p_x, simultaneously.

Linear momentum operator and translation in 1 dimension

- Show that $p_x = \frac{1}{i}\frac{\partial}{\partial x}$ is the generator for the translation; $U(\Delta x)\psi(x) = \psi(x + \Delta x)$, where $U(\Delta x) = e^{+ip_x \Delta x}$.
- Show that $[x, p_x] = i$.
- Show that $U^{-1}(\Delta x) = U^\dagger(\Delta x) = U(-\Delta x)$.
- Show that $U(\Delta x)p_x U^\dagger(\Delta x) = p_x$.
- Show that $U(\Delta x)xU^\dagger(\Delta x) = x + \Delta x$.
- Show that for a free particle($H = \frac{p_x^2}{2m}$) moving along the x-axis, $[H, p_x] = 0$ and therefore $U(\Delta x)p_x U^\dagger(\Delta x) = p_x$.

Linear momentum operator and translation in 3 dimensions

- Show that $p_i = \frac{1}{i}\frac{\partial}{\partial x_i}$'s are the generators for the 3-d translation; $U(\Delta\boldsymbol{x})\psi(\boldsymbol{x}) = \psi(\boldsymbol{x} + \Delta\boldsymbol{x})$, where $U(\Delta\boldsymbol{x}) = e^{+i\boldsymbol{p}\cdot\Delta\boldsymbol{x}}$.

- Show that $[x_i, p_j] = i\delta_{ij}$.

- Show that $U^{-1}(\Delta\boldsymbol{x}) = U^{\dagger}(\Delta\boldsymbol{x}) = U(-\Delta\boldsymbol{x})$.

- Show that $U(\Delta\boldsymbol{x})\boldsymbol{p}U^{\dagger}(\Delta\boldsymbol{x}) = \boldsymbol{p}$.

- Show that $U(\Delta\boldsymbol{x})\boldsymbol{x}U^{\dagger}(\Delta\boldsymbol{x}) = \boldsymbol{x} + \Delta\boldsymbol{x}$.

- Show that for a free particle($H = \frac{p_x^2}{2m}$) moving along the x-axis, $[H, p_x] = 0$ and therefore $U(\Delta\boldsymbol{x})p_xU^{\dagger}(\Delta\boldsymbol{x}) = p_x$.

Angular momentum operator and rotation in 3 dimensions

- Show that the rotation along the $z-$axis by an angle ϕ to the function $\psi(x, y)$ is
$\psi(x, y) \rightarrow R\psi(x, y) = \psi(x\cos\phi - y\sin\phi, y\cos\phi + x\sin\phi)$.

- Show that, as $\phi \rightarrow 0$,
$\psi(x\cos\phi - y\sin\phi, y\cos\phi + x\sin\phi) \rightarrow \psi(x - y\phi, y + x\phi) \rightarrow \left[1 + \phi\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi(x, y) = (1 + i\phi L_z)\,\psi(x, y)$, where
$L_z = xp_y - yp_x = \frac{1}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$.

- Using $\lim_{n\rightarrow\infty}\left(1 + \frac{\phi}{n}S\right)^n = e^{\phi S}$, show that
$R\psi(x, y) = \psi(x - y\phi, y + x\phi) = e^{+i\phi L_z}\psi(x, y)$.

- Show that the angular momentum operators L_i, $i = 1, 2, 3$ are generators of rotation.
- Show that the angular momentum operators L_i satisfies the Lie algebra $[L_i.L_j] = i\epsilon_{ijk}L_k$.
- Show that in the Cartesian coordinate(representation), where $\psi(x, y, z) = (x, y, z)^T$, the three generators are

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

- Show that the angular momentum operators L_i, $i = 1, 2, 3$ have three distinctive eigenvalues -1, 0, and $+1$.

Rotation and SU(2)

- Show that SU(n) complex matrices have $n^2 - 1$ generators.
- Show that Pauli matrices are a set of generators for $SU(2)$.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Show that $\mathrm{Tr}(\sigma^i \sigma^j) = 2\delta_{ij}$; $\lambda = 2$.
- Show that the structure constant is $c_{ijk} = 2\epsilon^{ijk}$; $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$.
- Show that the Pauli matrices are Hermitian, traceless, and $\mathrm{Det}(\sigma^i) = -1$.

- Show that $S_i = \frac{1}{2}\sigma_i$ satisfies the Lie algebra for the angular moementum; $[S_i, S_j] = i\epsilon_{ijk}S_k$.
- Show that $(\boldsymbol{\sigma} \cdot \boldsymbol{a})^2 = \boldsymbol{a}^2 = \sum_i a_i^2$, where $\boldsymbol{a} = (a^1, a^2, a^3)$ is a real vector.
- Show that $(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})^{2n} = 1$, where $\hat{\boldsymbol{n}}^2 = 1$.
- Show that $(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})^{2n+1} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$.
- Show that $U = e^{i\frac{\phi}{2}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}} = e^{i\phi\boldsymbol{S}\cdot\hat{\boldsymbol{n}}}$ produces the rotation along the axis $\hat{\boldsymbol{n}}$ by an angle ϕ.
- Show that $U = e^{i\frac{\phi}{2}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}} = 1\ \cos\frac{\phi}{2} + i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sin\frac{\phi}{2}$.

$$= \begin{pmatrix} \cos\frac{\phi}{2} + i\hat{n}_3\sin\frac{\phi}{2} & i\left(\hat{n}_1 - i\hat{n}_2\right)\sin\frac{\phi}{2} \\ i\left(\hat{n}_1 + i\hat{n}_2\right)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} - i\hat{n}_3\sin\frac{\phi}{2} \end{pmatrix}$$

Ladder operator approach: Consider the angular momentum operators. They satisfy the following Lie algebra $[J_i, J_j] = i\epsilon_{ijk} J_k$.

- Show that $[A, B^2] = [A, B]B + B[A, B] \quad \forall A,\ B$.
- Show that $[J_1, J_2^2] = +i(J_2 J_3 + J_3 J_2)$.
- Show that $[J_1, J_3^2] = -i(J_2 J_3 + J_3 J_2)$.
- Show that $[J_i, \boldsymbol{J}^2] = 0$, where $\boldsymbol{J}^2 = J_1^2 + J_2^2 + J_3^2$.
- Show that $\boldsymbol{J}^2 = \frac{1}{2}\left(J_+ J_- + J_- J_+\right) + J_z^2$
- Defining $J_\pm = J_1 \pm i J_2$, show that $[\boldsymbol{J}^2, J_\pm] = 0$, $[J_z, J_\pm] = \pm J_\pm$, and $[J_+, J_-] = 2J_z$.

Because $[\boldsymbol{J}^2, J_z] = 0$, we may choose a representation $|\lambda m\rangle$, where $J_z|\lambda m\rangle = m|\lambda m\rangle$ and $\boldsymbol{J}^2|\lambda m\rangle = \lambda|\lambda m\rangle$.

- Show that $J_i^\dagger = J_i$.
- Show that $J_\pm^\dagger = J_\mp$.
- Show that $\langle\psi|AB|\psi\rangle = \langle\psi|BA|\psi\rangle$ if $B = A^\dagger$.
- Show that $J_z J_\pm|\lambda m\rangle = (m \pm 1)|\lambda m\rangle$. Thus $J_\pm \propto |jm \pm 1\rangle$.
- Using $J_1^2 + J_2^2 = \boldsymbol{J}^2 - J_3^2$, show that $\lambda \geq m^2$, where $\boldsymbol{J}^2|\lambda m\rangle = \lambda|\lambda m\rangle$.

- Show that $\boldsymbol{J}^2 = J_\mp J_\pm + J_3(J_3 \pm 1)$.
- If $j = \mathrm{Max}[m]$, $J_+|\lambda j\rangle = 0$. Using the condition, show that $\lambda = j(j+1)$. Hint: Calculate $J_- J_+|\lambda j\rangle = 0$.

From now on, we replace the λ by j.

- If $j' = \mathrm{Min}[m]$, $J_-|jj'\rangle = 0$. Using the condition, show that $j' = -j$. Hint: Calculate $J_+ J_-|jj'\rangle = 0$.
- Show that there are $2j+1$ states $|jm\rangle$; $m = -j, -j+1, \cdots, j$.

Homework set 1: (due: Sep 18, 2004)

1. (4.1.2) Show that rotations about the $z-$axis form a subgroup of SO(3). Show that this group is not an invarinat subgroup of SO(3).

2. (4.1.5) A subgroup H of G has elements h_i. Let $x \in G$ and $x \notin H$. Show that the conjuagate subgroup $xHx^{-1} = \{xh_ix^{-1} \mid i = 1, 2, \cdots\}$ satisfies the four group postulates and therefore is a group.

3. (4.2.2) Prove that the general form of 2×2 unitary, unimodular matrix is $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ with $aa^* + bb^* = 1$.

4. Based on the result, show the parametrization

$$\begin{pmatrix} \cos\frac{\phi}{2} + i\hat{n}_3 \sin\frac{\phi}{2} & i\left(\hat{n}_1 - i\hat{n}_2\right)\sin\frac{\phi}{2} \\ i\left(\hat{n}_1 + i\hat{n}_2\right)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} - i\hat{n}_3 \sin\frac{\phi}{2} \end{pmatrix}$$ is equivalent to

$$\begin{pmatrix} e^{i\xi}\cos\eta & e^{i\zeta}\sin\eta \\ -e^{-i\zeta}\sin\eta & e^{-i\xi}\cos\eta \end{pmatrix}$$ and covers all possible 3-d rotation.

5. Show that $J_\mp J_\pm |jm\rangle = [j(j+1) - m(m \pm 1)]\,|jm \pm 1\rangle = (j \mp m)(j \pm m + 1)|jm \pm 1\rangle$

6. Show that $J_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm +1)}|jm \pm 1\rangle$

7. Consider a SU(2) group. Choosing the generators as one half of the Pauli matrices, show that

$$S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

8. Consider a Lorentz boost $\begin{pmatrix} t' \\ x' \end{pmatrix} = L \begin{pmatrix} t \\ x \end{pmatrix}$, where

$L = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$. Show that the boost matrix can be expressed as $L = e^{\alpha\sigma_1} = 1\ \cosh\alpha + \sigma_1\ \sinh\alpha$, where

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Chaper 6
Functions of Complex Variables

Complex Number

$$C \equiv \{z = x + iy \mid x, y \in R \text{ and } i = \sqrt{-1}\}$$

- Show that C is closed under multiplication.
- Show that $x + iy = r(\cos\theta + i\sin\theta)$, where $\cos\theta = x/r$, $\sin\theta = y/r$, and $r = |z| \equiv \sqrt{x^2 + y^2}$.
- Defining $z^* = \mathrm{Re}(z) - i\mathrm{Im}(z)$, show that $zz^* = |z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 = r^2$.

Complex Number and 2-d vector

$$\begin{aligned} z &= x + iy = re^{i\theta},\ r = \sqrt{x^2 + y^2} \\ x &= r\cos\theta,\ y = r\sin\theta \\ \boldsymbol{r} &= \hat{\mathbf{x}}x + \hat{\mathbf{y}}y : 1 - \text{to} - 1 \text{ correspondence} \end{aligned}$$

- Show that $z^{-1} \in C \; \forall \; z \in C - \{0\}$ and $z^{-1} = z^*/|z|^2$.
- Show that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, where $\theta = \arg(z)$ and $z = |z|(\cos\theta + i\sin\theta)$.
- Show that $|z| \geq |\mathrm{Re}(z)| \geq \mathrm{Re}(z)$.
- Show that $|z| \geq |\mathrm{Im}(z)| \geq \mathrm{Im}(z)$.
- Show that $|z_1 z_2| \geq |\mathrm{Re}(z_1 z_2)|, |\mathrm{Im}(z_1 z_2)|$.
- Show that $|z_1 z_2| \geq |\mathrm{Re}(z_1 z_2^*)|, |\mathrm{Im}(z_1 z_2^*)|$.
- Show that $|z| \pm \mathrm{Re}(z) \geq 0$.
- Show that $|z| \pm \mathrm{Im}(z) \geq 0$.

Schwarz inequality

- Show that $|x+y| \leq |x| + |y| \; \forall \; x, y \in R$. Hint: $|x| \geq \pm x$.
- Show that $|x| - |y| \leq |x+y| \; \forall \; x, y \in R$. Hint: $|x| \geq \pm x$.
- Therefore $|x| - |y| \leq |x+y| \leq |x| + |y| \; \forall \; x, y \in R$.
- Show that $|z|^2 \geq 0$. Thus $|\lambda z_1 + z_2|^2 \geq 0$.
- Choose real λ and show that $\mathrm{Re}(z_1 z_2^*) \leq |z_1 z_2^*| = |z_1||z_2|$.
- Show that $|z_1||z_2| \geq \pm \mathrm{Re}(z_1 z_2^*)$ leads to $|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2| \; \forall \; z_1, z_2 \in C$.
- Interpret this result in terms of vectors.

$e^{i\theta}$

- Show that $i^2 = -1 \to i^{2n} = (-1)^n,\ i^{2n+1} = i(-1)^n$.
- Show that $\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n}$.
- Show that $\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$.
- Show that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.
- Show that $e^{i\theta} = \cos\theta + i\sin\theta$
- Show that $|e^{i\theta}| = 1$.

De Moivre's Formula

- Show that $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$.
- Show that $z^n = r^n e^{in\theta}$
- Show that $(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}$.
- Show that $\cos n\theta = \sum_{k=0}^{2k\le n} \frac{(-1)^k n!}{(2k)!(n-2k)!} \cos^{n-2k}\theta \sin^{2k}\theta$.
- Show that
  $\sin n\theta = \sum_{k=0}^{2k+1\le n} \frac{(-1)^k n!}{(2k+1)!(n-2k-1)!} \cos^{n-2k-1}\theta \sin^{2k+1}\theta$.
- Prove the De Moivre's Formula
  $e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$.

Problem 6.1.6

- Show that $\sum_{n=0}^{N-1} ar^{n-1} = \frac{a(1-r^N)}{1-r}$.
- Show that

$$\sum_{n=0}^{N-1}(e^{i\theta})^n = \frac{1-e^{iN\theta}}{1-e^{i\theta}} = e^{i\frac{(N-1)\theta}{2}} \times \frac{e^{i\frac{N\theta}{2}} - e^{-i\frac{N\theta}{2}}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}$$

$$= e^{i\frac{(N-1)\theta}{2}} \times \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}$$

- Show that $\sum_{n=0}^{N-1} \cos n\theta = \cos\frac{(N-1)\theta}{2} \times \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}$
- Show that $\sum_{n=0}^{N-1} \sin n\theta = \sin\frac{(N-1)\theta}{2} \times \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}$

Single-slit diffraction

$$
\begin{aligned}
E &= \frac{1}{N}\sum_{k=1}^{N} E_k \rightarrow E_0 \sin\omega t \text{ if } \theta = 0 \\
E_k &= \frac{E_0}{N}\sin\left(\omega t + \frac{2\pi a \sin\theta}{\lambda}\frac{k}{N}\right) = E_0 \mathrm{Im}\, e^{i\left(\omega t + \frac{2\pi a \sin\theta}{\lambda}\frac{k}{N}\right)} \\
E &= \frac{E_0}{N}\mathrm{Im}\sum_{k=1}^{N} e^{i\left(\omega t + \frac{2\pi a \sin\theta}{\lambda}\frac{k}{N}\right)} \\
&= \frac{E_0}{N}\mathrm{Im}\left[e^{i\omega t}\sum_{k=1}^{N}\left(e^{i\frac{2\pi \sin\theta}{\lambda}\cdot\frac{a}{N}}\right)^k\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{E_0}{N}\mathrm{Im}\left[e^{i\omega t}\frac{1-e^{i\frac{2\pi a\sin\theta}{\lambda}}}{1-e^{i\frac{2\pi}{\lambda}\cdot\frac{a\sin\theta}{N}}}\right] \\
&= \frac{E_0}{N}\mathrm{Im}\left[e^{i\omega t}\frac{e^{i\frac{\pi a\sin\theta}{\lambda}}}{e^{i\frac{\pi}{\lambda}\cdot\frac{a\sin\theta}{N}}}\frac{\sin\left(\frac{\pi a\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi a\sin\theta}{\lambda N}\right)}\right] \\
&= \frac{E_0}{N}\frac{\sin\left(\frac{\pi a\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi a\sin\theta}{\lambda N}\right)}\mathrm{Im}e^{i\left(\omega t+\frac{\pi a\sin\theta}{\lambda}\frac{N-1}{N}\right)} \\
\lim_{N\to\infty} E &= E_0 \times \frac{\sin\alpha}{\alpha} \times \sin\left(\omega t + \frac{\pi a\sin\theta}{\lambda}\right), \ \alpha = \frac{\pi a\sin\theta}{\lambda} \\
\frac{I}{I_m} &= \frac{E^2|_{\mathrm{avg}}}{E_0^2|_{\mathrm{avg}}} = \left(\frac{\sin\alpha}{\alpha}\right)^2
\end{aligned}
$$

Assumimg $|p| < 1$ show the following formulas.

- $\sum_{n=0}^{\infty} p^n \cos n\theta = \mathrm{Re} \sum_{n=0}^{\infty} p^n e^{in\theta}$.
- $\sum_{n=0}^{\infty} p^n \sin n\theta = \mathrm{Im} \sum_{n=0}^{\infty} p^n e^{in\theta}$.
- $\sum_{n=0}^{\infty} p^n e^{in\theta} = \frac{1}{1-pe^{in\theta}}$.
- $\mathrm{Re} \frac{1}{1-pe^{in\theta}} = \frac{1-p\cos\theta}{1-2p\cos\theta+p^2}$.
- $\mathrm{Im} \frac{1}{1-pe^{in\theta}} = \frac{p\sin\theta}{1-2p\cos\theta+p^2}$.
- $\sum_{n=0}^{\infty} p^n \cos n\theta = \frac{1-p\cos\theta}{1-2p\cos\theta+p^2}$.
- $\sum_{n=0}^{\infty} p^n \sin n\theta = \frac{p\sin\theta}{1-2p\cos\theta+p^2}$.

Prove the following formulas.

- $e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}(\cos y + i \sin y)$.
- $e^{-z} = e^{-x}e^{-iy} = e^{-x}(\cos y - i \sin y)$.
- $e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix} = e^{-y}(\cos x + i \sin x)$.
- $e^{-iz} = e^{-i(x+iy)} = e^{y}e^{-ix} = e^{y}(\cos x - i \sin x)$.
- $\cos iz = \frac{e^{i(iz)}+e^{-i(iz)}}{2} = \cosh z$.
- $\sin iz = \frac{e^{i(iz)}-e^{-i(iz)}}{2i} = i \sinh z$.
- $\cosh iz = \frac{e^{(iz)}+e^{-(iz)}}{2} = \cos z$.
- $\sinh iz = \frac{e^{(iz)}-e^{-(iz)}}{2} = i \sin z$.

Prove the following formulas.

- $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$.
- $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$.
- $\cosh^2 y - \sinh^2 y = 1$.
- $|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y$.
- $|\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y$.
- $\forall x \in R \; \sin^2 x \leq 1$.
- $\forall x \;\; |\sin(x+iy)|^2 \geq 1$ if $|y| > \ln(1+\sqrt{2})$.
- $|\sin z| \geq |\sin x|$.
- $|\cos z| \geq |\cos x|$.

Prove the following formulas.

- $\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$.
- $\cosh(x+iy) = \cosh x \cos y + i \sin x \sinh y$.
- $|\sinh(x+iy)|^2 = \sinh^2 x + \sin^2 y$.
- $|\cosh(x+iy)|^2 = \cosh^2 x + \cos^2 y$.

Prove the following formulas.

- $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.
- $\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1$.
- $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$.
- $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$.
- $\sinh x = 2\sinh\frac{x}{2}\cosh\frac{x}{2}$.
- $\cosh x = 2\sinh^2\frac{x}{2} + 1 = 2\cosh^2\frac{x}{2} - 1$.
- $\tanh\frac{x}{2} = \frac{\cosh x-1}{\sinh x}$.
- $\tanh^2\frac{x}{2} = \frac{\cosh x-1}{\cosh x+1}$.

Prove the following formulas.

- $|(\cosh z) \pm 1|^2 = (\cosh x \cos y \pm 1)^2 + (\sinh x \sin y)^2 = (\cosh x \pm \cos y)^2$.

- $[(\cosh z) \pm 1][(\cosh z) \mp 1]^* = (\sinh x \mp i \sin y)^2$.

- $\tanh^2 \frac{z}{2} = \left(\frac{\sinh x + i \sin y}{\cosh x + \cos y}\right)^2$.

- $\coth^2 \frac{z}{2} = \left(\frac{\sinh x - i \sin y}{\cosh x - \cos y}\right)^2$.

Square root Solve $z^2 = Re^{i\Theta}$

$$\begin{aligned} z &= re^{i\theta} \\ z^2 &= r^2 e^{in\theta} = Re^{i(\Theta+2k\pi)} \\ r &= \sqrt{R} \\ \theta &= \frac{\Theta}{2}, \ \frac{\Theta}{2} + \pi \\ z_1 &= \sqrt{R}e^{i\frac{\Theta}{2}}, z_2 = \sqrt{R}e^{i\left(\frac{\Theta}{2}+\pi\right)} \end{aligned}$$

- Solve $z^2 = 1$.
- Solve $z^2 = -1$.
- Solve $z^2 = i$.

N-th root

Solve $z^n = Re^{i\Theta}$

$$\begin{aligned} z &= re^{i\theta} \\ z^n &= r^n e^{in\theta} = Re^{i(\Theta+2k\pi)} \\ r &= R^{1/n} \\ \theta &= \frac{\Theta + 2k\pi}{n}, \ k = 0, 1, \cdots, n-1 \end{aligned}$$

- Solve $z^3 = 1$.
- Solve $z^4 = -1$.
- Solve $z^5 = i$.

Logarithm

$$
\begin{aligned}
\ln z &= \ln\left[re^{i(\theta+2n\pi)}\right] \\
&= \ln r + i(\theta + 2n\pi) \rightarrow \text{multi} - \text{valued} \\
&\quad \text{cut } (\theta_0 - \pi < \theta < \theta_0 + \pi) \text{ is needed} \\
&= \ln r \rightarrow \text{principal value}
\end{aligned}
$$

$$
\begin{aligned}
e^{\ln z} &= e^{\ln\left[re^{i(\theta+2n\pi)}\right]} \\
&= e^{\ln r + i(\theta+2n\pi)} \\
&= e^{\ln r} e^{i(\theta+2n\pi)} \\
&= re^{i\theta} = z
\end{aligned}
$$

Cauchy-Riemann condition If $f(z)$ is differentiable

$$\begin{aligned} f'(z) &= \frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \\ &= \frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(z+\delta x) - f(z)}{\delta x} \\ &= \frac{\partial f}{\partial (iy)} = \lim_{\delta y \to 0} \frac{f(z+i\delta y) - f(z)}{i\delta y} \end{aligned}$$

What happens if

$$\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial (iy)}?$$

Laplace equation and Harmonic function: Consider a differentiable function $f(z)$;

$$f(z) = u(z) + iv(z), \ f_a \equiv \frac{\partial f}{\partial a}$$

- Show that $u_x = v_y, \quad u_y = -v_x$.
- Show that $u_{xx} = v_{yx} = v_{xy} = -u_{yy} \rightarrow \nabla^2 u = 0$
- $v_{yy} = u_{xy} = v_{yx} = -v_{xx} \rightarrow \nabla^2 v = 0$
- u and v are harmonic functions; solutions to 2-d Laplace equation; potential function.
- Show that the two 2-dimensional vectors (u_x, v_y) and (u_y, v_x) are orthogonal; $u_x u_y + v_x v_y = 0$.

Analytic function: A function is analytic at $z = z_0$; If the function is differentiable at $z = z_0$ and in some small region around z_0. **Entire function**: Analytic everywhere.

- Show that $f(z) = z$ is analytic. $(u = x,\ v = y)$
- Show that $f(z) = \mathrm{Re}(z)$ is not analytic. $(u = x,\ v = 0)$
- Show that $f(z) = \mathrm{Im}(z)$ is not analytic. $(u = 0,\ v = y)$
- Show that $f(z) = z^*$ is not analytic. $(u = x,\ v = -y)$
- Show that $f(z) = z^2$ is analytic. $(u = x^2 - y^2,\ v = 2xy)$
- Show that $f(z) = |z|^2 = zz^*$ is not analytic. $(u = x^2 + y^2,\ v = 0)$

- $f(z) = u + iv$ and $g(z) = u' + iv'$ are analytic. Using Cauchy-Rieman conditions for $f(z)$ and $g(z)$, show that $h(z) = f(z) + g(z) = U + iV$ is also analytic.
  $(U = u + u',\ V = v + v')$
- $f(z) = u + iv$ and $g(z) = u' + iv'$ are analytic. Using Cauchy-Rieman conditions for $f(z)$ and $g(z)$, show that $h(z) = f(z)g(z) = U + iV$ is also analytic.
  $(U = uu' - vv',\ V = uv' + vu')$
- Using above result, show that $[f(z)]^n$ is analytic if $f(z)$ is analytic.
- Show that z^n is analytic.

Constant electric field along the $\boldsymbol{x}$-axis

$$
\begin{aligned}
\Phi(z) &= \phi(z) + i\psi(z) = -E_0 z \\
\boldsymbol{E} &= -\nabla\phi = -\phi_x\hat{\mathbf{x}} - \phi_y\hat{\mathbf{y}}, \quad \nabla^2\phi = -\nabla\cdot\boldsymbol{E} = 0 \\
E_x &= -\phi_x, \quad E_y = -\phi_y
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\Phi(z)}{dz} &= -E_0 \\
&= \phi_x + i\psi_x = \phi_x - i\phi_y = -E_x + iE_y \\
E_x &= -\mathrm{Re}\frac{d\Phi(z)}{dz} = E_0 \\
E_y &= +\mathrm{Im}\frac{d\Phi(z)}{dz} = 0
\end{aligned}
$$

$\boldsymbol{E}$ field along a long charged wire

$$\begin{aligned}
\oint \boldsymbol{E}\cdot d\boldsymbol{A} &= \frac{Q}{\varepsilon_0} \leftarrow Q = \lambda L \\
\boldsymbol{E} &= \frac{\lambda}{2\pi\varepsilon_0}\frac{\boldsymbol{r}}{r^2},\ E_x = \frac{\lambda}{2\pi\varepsilon_0}\frac{x}{r^2},\ E_y = \frac{\lambda}{2\pi\varepsilon_0}\frac{y}{r^2} \\
\frac{d\Phi(z)}{dz} &= -E_x + iE_y = \frac{\lambda}{2\pi\varepsilon_0}\frac{-z^*}{|z|^2} = -\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{z} \\
\Phi(z) &= -\frac{\lambda}{2\pi\varepsilon_0}\int\frac{dz}{z} = -\frac{\lambda}{2\pi\varepsilon_0}\ln z \\
&= -\frac{\lambda}{2\pi\varepsilon_0}(\ln r + i\theta) \\
\phi &= \mathrm{Re}\Phi = -\frac{\lambda}{2\pi\varepsilon_0}\ln r
\end{aligned}$$

$\boldsymbol{B}$ field around a long wire

$$
\begin{aligned}
\Phi(z) &= \phi(z) + i\psi(z) = \phi + iA(z) \\
\boldsymbol{B} &= \nabla \times \boldsymbol{A} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y, \quad \boldsymbol{A} = \hat{\mathbf{z}}A(x,y) \\
\nabla \cdot \boldsymbol{B} &= 0 = \nabla(\nabla \cdot A) - \nabla^2 \boldsymbol{A} = 0 \to \nabla^2 A = 0 \\
B_x &= \frac{\partial A}{\partial y} = -\frac{\mu_0 I}{2\pi}\frac{y}{r^2}, \quad B_y = -\frac{\partial A}{\partial x} = \frac{\mu_0 I}{2\pi}\frac{x}{r^2} \\
\frac{d\Phi(z)}{dz} &= \frac{\partial A}{\partial y} + i\frac{\partial A}{\partial x} = B_x - iB_y \\
&= \frac{\mu_0 I}{2\pi}\frac{(-y - ix)}{r^2} = -i\frac{\mu_0 I}{2\pi}\frac{x - iy}{r^2} = -i\frac{\mu_0 I}{2\pi z} \\
\Phi &= -i\frac{\mu_0 I}{2\pi}\ln z, \quad A = \mathrm{Im}\Phi = -\frac{\mu_0 I}{2\pi}\ln r
\end{aligned}
$$

A stupid way to show that z^n is analytic

$$
\begin{aligned}
z^n &= (x+iy)^n = u + iv \\
u &= \sum_{k=0}^{2k\le n} \frac{(-1)^k n!}{(2k)!(n-2k)!} x^{n-2k} y^{2k} \\
v &= \sum_{k=0}^{2k+1\le n} \frac{(-1)^k n!}{(2k+1)!(n-2k-1)!} x^{n-2k-1} y^{2k+1} \\
u_x &= v_y = \sum_{k=0}^{2k+1\le n} \frac{(-1)^k n!}{(2k)!(n-2k-1)!} x^{n-2k-1} y^{2k} \\
u_y &= -v_x = \sum_{k=0}^{2k\le n} \frac{(-1)^k n!}{(2k-1)!(n-2k)!} x^{n-2k} y^{2k-1}
\end{aligned}
$$

z^* is NOT differentiable

$$\begin{aligned} z^* &= x - iy = u + iv \;\rightarrow u = x,\; v = -y \\ u_x &= 1 \neq v_y = -1, \quad u_y = -v_x = 0 \end{aligned}$$

$1/z$

$$
\begin{aligned}
\frac{1}{z} &= \frac{z^*}{|z|^2} = u + iv, \quad u = \frac{x}{x^2+y^2}, \; v = -\frac{y}{x^2+y^2} \\
u_x &= \frac{-2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\
v_y &= \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \rightarrow u_x = v_y \\
u_y &= \frac{-2xy}{(x^2+y^2)^2} \\
v_x &= \frac{2xy}{(x^2+y^2)^2} \rightarrow u_y = -v_x
\end{aligned}
$$

However, the fuction is not defined at $z = 0$.

Elementary functions Explain why the following functions are analytic and prove the equalities.

$$\begin{aligned}
z^n &= \text{differentiable} \\
e^z &= \sum_{k=0}^{\infty} \frac{z^n}{n!} \\
\sin z &= \sum_{k=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\
\cos z &= \sum_{k=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\
\ln(1+z) &= \sum_{k=0}^{\infty} (-1)^n \frac{z^n}{n}, \quad |z| < 1
\end{aligned}$$

Prove

$$
\begin{aligned}
\frac{d}{dz} z^n &= n z^{n-1} \\
\frac{d}{dz} e^z &= e^z \\
\frac{d}{dz} \sin z &= \cos z \\
\frac{d}{dz} \cos z &= -\sin z \\
\frac{d}{dz} \ln(1+z) &= \frac{1}{1+z}, \ |z| < 1
\end{aligned}
$$

Homework set 2: (due: Sep. 25, 2004)

1. (6.1.14) Find all the zeros of (a) $\sin z$, (b) $\cos z$, (c) $\sinh z$, (d) $\cosh z$.

2. (6.1.15) Show that

   (a) $\sin^{-1} z = -i \ln\left(iz \pm \sqrt{1 - z^2}\right)$

   (b) $\sinh^{-1} z = \ln\left(z \pm \sqrt{z^2 + 1}\right)$

   (c) $\cos^{-1} z = -i \ln\left(iz \pm \sqrt{1 - z^2}\right)$

   (d) $\cosh^{-1} z = \ln\left(z \pm \sqrt{z^2 - 1}\right)$

   (e) $\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right)$

   (f) $\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$.

3. (6.1.20) Show that

(a) $e^{\ln z}$ always equals z.

(b) $\ln e^z$ does not always equals z.

4. (6.2.3) Having shown that the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $w(z)$ each satisfy Laplace's equation, show that $u(x, y)$ and $v(x, y)$ cannot both have either a maximum or a minimum in the interior of any region in which $w(x, y)$ is analytic. They can have saddle points.

5. (6.2.4) Let $A = w_{xx}$, $B = w_{xy}$, and $C = w_{yy}$. From the calculus of functions of two variables, $w(x, y)$, we have a saddle point if $B^2 - AC > 0$. With $f(z) = u(x, y) + iv(x, y)$, apply the Cauchy-Riemann conditions and show that both

$u(x, y)$ and $v(x, y)$ do not have a maximum or a minimum in a finite region of the complex plain.

6. (6.2.7) The function $f(z) = u(x, y) + iv(x, y)$ is analytic. Show that $[f(z^*)]^*$ is also analytic.

7. (6.2.8) A proof of the Schwarz inequality involves minimizing an expression

$$f = \psi_{aa} + \lambda^* \psi_{ab} + \lambda \psi_{ab}^* + \lambda \psi_{bb} \geq 0.$$

The ψ are integrals of products of functions; ψ_{aa} and ψ_{bb} are real, ψ_{ab} is complex, and λ is a complex parameter.

- Differentiate the preceding expression with respect to λ^*, treating λ as an independent parameter, independent of λ^*. Show that setting the derivative $\partial f / \partial \lambda^*$ equal to zero yields $\lambda = -\psi^*_{ab}/\psi_{bb}$.

- Show that $\partial f/\partial\lambda = 0$ leads to the same result.
- Let $\lambda = x + iy$, $\lambda^* = x - iy$. Set the x and y derivatives equal to zero and show that again $\lambda = -\psi^*_{ab}/\psi_{bb}$.

6.3 Cauchy's Integral Theorem

Integral exists if its value is independent of the path

$$\begin{aligned}
\int_{z_1}^{z_2} dz f(z) &= \int_{x_1+iy_1}^{x_2+iy_2} [u+iv][dx+idy] \\
&= \int_{x_1+iy_1}^{x_2+iy_2} [udx - vdy] \\
&+ i\int_{x_1+iy_1}^{x_2+iy_2} [vdx + udy] \\
&= F(z_2) - F(z_1)
\end{aligned}$$

Cauchy integral for powers: Consider a path C on a circle of radius r, where the center is the origin. We integrate z^n over the circle from $z = r$ through $z = re^{i2\pi}$.

- Show that the point on the path is $z = re^{i\theta}$, and $dz = ire^{i\theta} d\theta$, where the θ is the polar angle.
- Show that $\int_C z^n \, dz = 0 \; \forall$ integer $n \neq 1$.
- Show that $\int_C \frac{dz}{z} = 2\pi i$.
- Show that $\int_C \frac{dz}{z - z_0} = 0 \; \forall z_0$ such that $|z - z_0| > r$.
- Show that $\int_C dz (z - z_0)^n = 0 \; \forall n$ and $\forall z_0$ such that $|z - z_0| > r$.

If $|z| > 0$, z^n is analytic for any integer n

$$\begin{aligned}
C \quad &: \quad \text{circle of radius } r, \quad z = re^{i\theta}, \quad dz = ire^{i\theta} d\theta \\
n \quad &\neq \quad -1 \\
\int_C dz z^n \, dz \quad &= \quad ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta = \frac{r^{n+1}}{n+1} \left[e^{i(n+1)2\pi} - 1 \right] \\
&= \quad \left[\frac{z^{n+1}}{n+1} \right]_r^{re^{2\pi i}} = 0
\end{aligned}$$

$$\begin{aligned}
\int_C \frac{dz}{z} \, dz \quad &= \quad i \int_0^{2\pi} d\theta = 2\pi i \\
&= \quad \ln(re^{2\pi i}) - \ln r = 2\pi i
\end{aligned}$$

- Show that $\int_{x_1}^{x_2} f(x)dx = -\int_{x_2}^{x_1} f(x)dx$.
- Show that $\int_{x_1}^{x_2} f(x,y)dx = -\int_{x_2}^{x_1} f(x,y)dx$.
- Show that $\int_{z_1}^{z_2} f(z)dz = -\int_{z_2}^{z_1} f(z)dz$. Hint: Rewrite the integral in terms of the integrals of real variables x and y.
- For any contour encircling $z = 0$ once counterclockwise, $\frac{1}{2\pi i} \oint z^{m-n-1}\, dz = \delta_{mn}, \quad$ for integers m, n

Cauchy's integral theorem

If $f(z)$ is analytic and its partial derivatives are continuous throughout some simply connected region R, for every closed path C in R the integral of $f(z)$ around C vanishes or

$$\oint_C f(z)\, dz = 0$$

Proof using Stoke's theorem

$$\boldsymbol{V} = \hat{\mathbf{x}}V_x + \hat{\mathbf{y}}V_y$$

$$\oint_C \boldsymbol{V} \cdot d\boldsymbol{s} = \int_A \boldsymbol{\nabla} \times \boldsymbol{V} \cdot d\boldsymbol{A}$$

$$\oint_C (V_x dx + V_y dy) = \int_A \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) dx\ dy$$

$$\text{if } (V_x, V_y) = (u, -v), \quad \oint_C (u dx - v dy) = -\int_A (v_x + u_y) dx\ dy$$

$$\text{if } (V_x, V_y) = (v, u), \quad \oint_C (v dx + u dy) = \int_A (u_x - v_y) dx\ dy$$

It's like a conservative force

$$\begin{aligned}
\oint_C f(z)dz &= \oint_C (udx - vdy) + i\oint (udy + vdx) \\
&= \int_A \left[-(v_x + u_y) + i(u_x - v_y)\right] dx\ dy = 0 \\
&\leftarrow \mathrm{Cauchy - Riemann\ condition} \leftarrow \mathrm{analytic} \\
\oint_C f(z)dz &= \int_{z_1}^{z_2} f(z)dz + \int_{z_2}^{z_1} f(z)dz \\
&= 0 \\
\int_{z_1}^{z_2} f(z)dz &= F(z_2) - F(z_1)
\end{aligned}$$

$F(z)$ is like a potential

Simply connected region We take a contour C. If $f(z)$ is analytic $\forall z \in R$ such that $C = \partial R$, where ∂R is the boundary of R, R is simply connected. For all $C' \subset R$,
$\oint_C f(z)dz = \oint_{C'} f(z)dz = 0$
Multiply connected region $f(z)$ is analytic in R. If there is a path $C \subset R$ such that the region suroudeded by the C contains a region R', where $f(z)$ is not analytic, the region R is multiply connected.

Application Consider a contour C and C' which encircle the same region R' and $C \not\subset R'$ and $C' \not\subset R'$, where $f(z)$ is not analytic in R' and $f(z)$ is analytic in $(R')^c$. Choose two very close points $A, B \in C$ and $A', B' \in C$ and draw paths L from A to B and L' from A' to B'. Let L and L' approaches arbitrarily closely and $L \cap L' = \emptyset$.

- Show that $\int_{AB} f(z)dz \to 0$ and $\int_{A'B'} f(z)dz \to 0$.
- Show that $\oint_C f(z)dz = \oint_{C-AB} f(z)dz$ and $\oint_{C'} f(z)dz = \oint_{C'-A'B'} f(z)dz$.
- Show that $\oint_{AA'} f(z)dz = \oint_{B'B} f(z)dz = 0$.
- Draw a closed path in a simply connected region combining open curves passing A, B, A', B' along $C - AB$, $C' - A'B'$, L, and $L' \to -L$.

$$\oint_{C+C'+L-L} f(z)dz = \left[\int_C + \int_{-C'} + \int_L + \int_{-L}\right] f(z)dz = 0$$

$$\int_C f(z)dz - \int_{C'} f(z)dz + \int_L f(z)dz - \int_L f(z)dz = 0$$

$$\oint_C f(z)dz = \oint_{C'} f(z)dz$$

6.4 Cauchy's integral formula

If $f(z)$ is analytic in R within a boundary contour C

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz \quad = \quad \begin{cases} f(z_0) \leftarrow \text{ if } z_0 \text{ is enclosed by } C. \\ 0 \leftarrow \text{ if } z_0 \text{ is not enclosed by } C. \end{cases}$$

- Show that $\frac{1}{z^2+1}$ is analytic inside $\forall C$ which is parametrized as $C = \{z = re^{i\theta} \mid r < 1\}$. Therefore the region inside C is simply connected.

- Show that $\oint_C \frac{1}{z^2+1} = 0 \;\; \forall C$ which is parametrized as $C = \{z = re^{i\theta} \mid r < 1\}$.

If $f(z)$ is analytic in R within a boundary contour C

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}dz &= 0 \leftarrow \frac{f(z)}{z-z_0} \text{ is analytic inside } C_1 \\
\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}dz &= \oint_{z=z_0+re^{i\theta}}\frac{f(z)}{z-z_0}dz \\
&= \frac{1}{2\pi i}\lim_{r\to 0}\int\frac{f(z_0+re^{i\theta})}{re^{i\theta}}re^{i\theta}id\theta \\
&= \frac{1}{2\pi i}\lim_{r\to 0}f(z_0)\int_0^{2\pi}id\theta = f(z_0)
\end{aligned}$$

$\boldsymbol{n}$-th Derivatives If $f(z)$ is analytic in R within a boundary contour $C = \partial R$. $\forall w$ such that $w \in R$ and $w \notin C$ show that

$$f(w) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - w} dz.$$

$$\begin{aligned}
f'(w) &\equiv \lim_{\delta \to 0} \frac{f(w+\delta) - f(w)}{\delta} \\
&= \lim_{\delta \to 0} \frac{1}{2\pi i \delta} \left[\oint_C \frac{f(z)}{z - (w+\delta)} dz - \oint_C \frac{f(z)}{z - w} dz \right] \\
&= \lim_{\delta \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)}{[z - (w+\delta)][z - w]} dz \\
&= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - w)^2} dz.
\end{aligned}$$

$f(z)$ is analytic on and within a closed contour C.

- Show that
$$\oint_C \frac{f'(z)}{z - z_0} dz = 2\pi i f''(z_0).$$
- Show that
$$f''(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz \quad \forall z_0 \text{ enclosed by } C.$$
- Therefore
$$\oint_C \frac{f'(z)}{z - z_0} dz = \oint_C \frac{f(z)}{(z - z_0)^2} dz \quad \forall z_0 \text{ enclosed by } C.$$

***n*-th Derivatives** $f(z)$ is analytic on and within a closed contour C. $f^{(n)}$ is the $n-$th derivative of $f(z)$.

- Check if

$$\frac{0!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^1}dz = f(z_0).$$

- Assume

$$\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = f^{(n)}(z_0).$$

- Show that

$$\frac{(n+1)!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+2}}dz = f^{(n+1)}(z_0).$$

- Hint: Show that

$$f^{(n+1)}(z_0) = \lim_{\delta \to 0} \frac{f^{(n)}(z_0 + \delta) - f^{(n)}(z_0)}{\delta}$$

and use

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

You must know $(1/z^n)' = -n/z^{n+1}$ for $z \neq 0$.

- Therefore, by mathematical induction, $\forall n \geq 0$

$$\frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = f^{(n)}(z_0).$$

6.4.7 Legendre polynomial Now we know that for any analytic function $f(z)$ within the contour C surrounding z_0

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Show that Legendre's polynomial $P_n(x)$ is expressed as

$$\begin{aligned} P_n(x) &= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{(-1)^n}{2^n n!}\frac{n!}{2\pi i}\oint_C \frac{(1-z^2)^n}{(z-x)^{n+1}}dz \\ &= \frac{(-1)^n}{2^n}\cdot\frac{1}{2\pi i}\oint_C \frac{(1-z^2)^n}{(z-x)^{n+1}}dz \end{aligned}$$

where the contour encloses x once in a positive sense. This is called Schläfli's integral representation for the $P_n(x)$.

Legendre's integral representation Choose the contour C as a cicle around x with radius $\sqrt{1-x^2}$ so that $z = x + ie^{i\phi}\sqrt{1-x^2}$, where $0 < \phi < 2\pi$. Show that

- $dz = -\sqrt{1-x^2}e^{i\phi}d\phi = i(z-x)d\phi$.
- $z^2 - 1 = 2(z-x)\left(x + i\sqrt{1-x^2}\cos\phi\right)$.

$$\begin{aligned} P_n(x) &= \frac{1}{2^n \cdot 2\pi i}\oint_C \frac{(z^2-1)^n}{(z-x)^{n+1}}dz \\ &= \frac{1}{2\pi}\int_0^{2\pi}(x + i\sqrt{1-x^2}\cos\phi)^n d\phi, \\ P_n(\cos\theta) &= \frac{1}{\pi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^n d\phi. \end{aligned}$$

This is called Legendre's integral representation for the $P_n(x)$.

Generating function for the Legendre's polynomial Change the integration variable into $t = \cos\theta + i\sin\theta\cos\phi$.

- Show that t runs from $e^{-i\theta}$ to $e^{i\theta}$ and

$$d\phi = \frac{dt}{i\sin\theta\cos\theta} = \frac{dt}{i\sqrt{t^2 - 2t\cos\theta + 1}}$$

- Show that

$$P_n(\cos\theta) = \frac{1}{\pi i}\int_{e^{-i\theta}}^{e^{i\theta}} dz \frac{z^n}{\sqrt{z^2 - 2z\cos\theta + 1}}.$$

- Next We will show that

$$g(t, \cos\theta) \equiv \frac{1}{\sqrt{t^2 - 2t\cos\theta + 1}} = \sum_{n=0}^{\infty} t^n P_n(\cos\theta).$$

This is called Legendre's integral representation for the $P_n(x)$.

Consider $-1 < x < 1$, $0 < t < 1$, and $z \in C = \{e^{i\theta} \mid 0 \leq \theta < 2\pi\}$ so that C encloses x

- Show that the three points $-1, z, 1$ make a right triangle and the area is $|1 - z^2|/2 = \mathrm{Im}(z)$.
- Show that $|z - x| \geq \mathrm{Im}(z)$ and therefore $\left|\frac{t(z^2-1)}{2(z-x)}\right| < 1$.
- Using $P_n(x) = \frac{1}{2^n \cdot 2\pi i} \oint_C \frac{(z^2-1)^n}{(z-x)^{n+1}} dz$ and above results, show that the following infinite series is convergent as

$$\sum_{n=0}^{\infty} t^n P_n(x) \;=\; -\frac{1}{t\pi i} \oint_C \frac{dz}{(z - z_+)(z - z_-)}.$$

where $z_\pm = \frac{1}{t}\left(1 \pm \sqrt{1 - 2xt + t^2}\right)$.

- Show that only z_- is within the contour and the integral becomes

$$g(t,x) = \sum_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}.$$

We have derived the closed form of the generating function for Legendre's polynomials.

Homework set 3: (due: Oct. 9, 2004)

1. Using the Schwarz inequality to prove $\left|\int_C f(z)dz\right| \leq |f_{\max}|\, L$, where $|f(z)| \leq |f_{\max}|\ \forall z \in C$ and L is the length of the path C. We will make use of this result frequently.

2. We learned that z^* is not analytic. We will find the integral of z^* may depend on the path. Show that $\int_0^{1+i} z^* dz$ depends on the path. a) Integrate along $C_1 = t$ and then $C_2 = 1 + it$, where $0 < t < 1$. b) Integrate along $C_1 = it$ and then $C_2 = t + i$, where $0 < t < 1$.

3. a) Show that $\oint_C \frac{dz}{z(1+z)} = 0$ if C is $z = re^{i\theta},\ 0 < \theta < 2\pi$ and $r < 1$. b) Show that $\oint_C \frac{dz}{z(1+z)} = 2\pi i$ if C is $z = re^{i\theta},\ 0 < \theta < 2\pi$ and $r < 1$.

4. Show that

a) $P_n(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2 - 1)^\ell$ is the solution to the Legendre's differential equation

$$\frac{d}{dx}\left[(1 - x^2)\frac{d}{dx}P_\ell(x)\right] + \ell(\ell + 1)P_\ell(x) = 0.$$

b) Replacing $x = \cos\theta$, show that the Legendre's equation is equivalent to

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}P_\ell(\cos\theta)\right] = \ell(\ell + 1)P_\ell(\cos\theta).$$

c) Show that in the spherical coordinate system

$$\begin{aligned}
\boldsymbol{\nabla} &= \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \\
L_z &= -i\frac{\partial}{\partial \phi}, \\
\boldsymbol{L}^2 &= -\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial}{\partial \theta}\right] - \frac{1}{\sin^2\theta}\frac{\partial}{\partial \phi^2}.
\end{aligned}$$

and $P_n(\cos\theta)$ is the eigenfunction for the orbital angular momentum $j = \ell$ and $j_z = 0$.

5. Prove Morera's Theorem: If a function $f(z)$ is continuous in a simply connected region R and $\oint_C f(z)dz = 0 \ \forall$ closed contour C within R, then $f(z)$ is analytic throughout R.
6. Prove Cauchy's inequality: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic

and bounded, $|f(z)| \leq M$ on a circle of radius r about the origin, then

$$|a_n|r^n \leq M$$

gives upper bounds for the coefficients of its Taylor series expansion.

7. Prove Liouville's theorem: If $f(z)$ is analytic and bounded in the complex plain, it is a constant function.

8. Using Liouville's theorem, prove the fundamental theorem of algebra: Any poplynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ with $n > 0$ and $a_n \neq 0$ has n roots.

6.3 Laurent Expansion

Taylor Expansion If $f(z)$ is analytic inside the contour C

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{w-z} = \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{(w-z_0)-(z-z_0)} \\
&= \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{(w-z_0)\left[1-\frac{z-z_0}{w-z_0}\right]} \\
&= \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n dw \\
&= \sum_{n=0}^{\infty}(z-z_0)^n \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{(w-z_0)^{n+1}} \\
&= \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{n!} f^{(n)}(z_0)
\end{aligned}$$

- Show that $\ln(1+z) = -\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n}$.
- Show that $\forall m \in R$ and $|z| < 1$,

$$
\begin{aligned}
(1+z)^m &= 1 + mz + \frac{m(m-1)}{2 \cdot 1} z^2 + \frac{m(m-1)(m-2)}{3 \cdot 2 \cdot 1} z^2 + \cdots \\
&= \sum_{n=0}^{\infty} \left(\begin{array}{c} m \\ n \end{array} \right) z^n,
\end{aligned}
$$

where $\left(\begin{array}{c} m \\ n \end{array} \right)$ is $\frac{m!(m-n)!}{n!}$ generalized into the real numbers.

Schwarz reflection principle

- Consider a complex function $g(z) = (z - x_0)^n$, where x_0 and n are real numbers. Using binominal expansion generalized to real powers, show that $[g(z)]^* = (z^* - x_0)^n = g(z^*)$.
- Consider a function which is analytic around $x_0 \in R$. Show that the Talyor expansion near the point $f(z) = \sum_{n=0}^{\infty}(z - x_0)^n f^{(n)}(x_0)/n!$ exists.
- Show that if the function is real if z is real, then $f^{(n)}(x_0)$ is real $\forall n$ and, therefore, $[f(z)]^* = f(z^*)$.

Using Schwarz reflection principle,

- Show that $[e^z]^* = e^{z^*}$.
- Show that $[\sin z]^* = \sin(z^*)$.
- Show that $[\ln(1+z)]^* = \ln(1+z^*)$.

Analytic continuation $(1+z)^{-1}$ is NOT analytic at $z=-1$.

- Show that the series expansion $(1+z)^{-1} = \sum_{n=0}^{\infty}(-z)^n = 1 - z + z^2 - z^3 + \cdots$ converges for $|z| < 1$. Hint: Calculate $\sum_{n=0}^{N}(-1)^n z^n$ and take limit $n \to \infty$.
- Above expansion is around $z = 0$. We know the function is analytic $\forall z \neq -1$. Let us expand the function around $z_0 \neq -1$ as well as $z_0 \neq 0$.

$$\frac{1}{1+z} = \frac{1}{(1+z_0)+(z-z_0)} = \frac{1}{(1+z_0)\left[1+\frac{z-z_0}{1+z_0}\right]}$$

$$= \frac{1}{1+z_0}\sum_{n=0}^{\infty}\left(-\frac{z-z_0}{1+z_0}\right)^n$$

- Show the series converges if $|z-z_0| < |1+z_0|$.

Laurent expansion Even if $f(z)$ is singular at z_0, we can expand $f(z)$ in terms of $(z - z_0)^n$ in an analytic region between C_1 and C_2, where $f(z)$ is not analytic in R' such that $z_0 \in R'$ and C_2 encloses R'. A larger contour C_1 encloses both R' and C_2. If $f(z)$ is analytic in the region R between C_1 and C_2.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The series is called Laurent series. Let us derive the explicit form of the series.

derivation Let us evaluate the integral S_1 along the larger closed contour C_1. We choose $w \in C_1$ and $z \in R$ so that $|w - z_0| > |z - z_0| \to \frac{|z-z_0|}{|w-z_0|} < 1$. Show that the integral is expressed as a convergent power series;

$$
\begin{aligned}
S_1 &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)dw}{w - z} = \frac{1}{2\pi i}\oint_{C} \frac{f(w)dw}{(w - z_0) - (z - z_0)} \\
&= \frac{1}{2\pi i}\oint_{C} \frac{f(w)dw}{(w - z_0)\left[1 - \frac{z-z_0}{w-z_0}\right]} \\
&= \frac{1}{2\pi i}\oint_{C} \frac{f(w)}{(w - z_0)} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^n dw \\
&= \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i}\oint_{C} \frac{f(w)dw}{(w - z_0)^{n+1}}.
\end{aligned}
$$

Next, we evaluate the integral S_2 along the smaller closed contour C_2. We choose $w \in C_2$ and $z \in R$ so that $|z - z_0| > |w - z_0| \to \frac{|w-z_0|}{|z-z_0|} < 1$. Show that the integral is expressed as a convergent power series;

$$\begin{aligned}
S_2 &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)dw}{z-w} = \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{(z-z_0)-(w-z_0)} \\
&= \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{(z-z_0)\left[1-\frac{w-z_0}{z-z_0}\right]} \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\oint_C \frac{(w-z_0)^{n-1}}{(z-z_0)^{n+1}} f(w)dw \\
&= \sum_{n=-1}^{-\infty}(z-z_0)^n \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{(w-z_0)^{n+1}}
\end{aligned}$$

- Show that the sum $S_1 - S_2$ is the contour integral surrounding a simply connected region including z. Thus

$$f(z) = S_1 - S_2 = \frac{1}{2\pi i}\left[\oint_{C_1} \frac{f(w)dw}{w - z} - \oint_{C_2} \frac{f(w)dw}{w - z}\right].$$

- Therefore,

$$f(z) = \sum_{n=-\infty}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \oint_C \frac{f(w)dw}{(w - z_0)^{n+1}},$$

where the contour C is again enclosing multiply connected region including z_0 and between C_1 and C_2.

Example 6.5.1: The function $1/[z(1-z)]$ is not analytic at both $z=0$ and $z=1$. But the function is analytic elsewhere such as $0<|z|<1$. We want to find the Laurent expansion, for example, around $z=0$:

$$f(z)=\frac{1}{z(1-z)}=\sum_{n=-\infty}^{\infty} a_n(z-0)^n$$

Choosing the contour $C = \{w \mid 0 < |w| < 1\}$,

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\oint_C \frac{1}{w(1-w)}\frac{dw}{(w-0)^{n+1}} = \frac{1}{2\pi i}\oint_C \frac{dw}{w^{n+2}(1-w)} \\
&= \sum_{k=0}^{\infty}\frac{1}{2\pi i}\oint_C w^k \frac{dw}{w^{n+2}} = \sum_{k=0}^{\infty}\frac{1}{2\pi i}\oint_C \frac{dw}{w^{(n+1-k)+1}} \\
&= \sum_{k=0}^{\infty}\delta_{n+1,k} = \begin{cases} 1 \text{ if } n \geq -1 \\ 0 \text{ if } n < -1 \end{cases} \\
f(z) &= \frac{1}{z} + 1 + z + z^2 + \cdots
\end{aligned}$$

Taylor or Laurent

$$f(z) = \frac{1}{1-z}$$

for $|z| < 1$

$$f(z) = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n$$

for $|z| > 1$

$$f(z) = \frac{1}{1-z} = \frac{\frac{1}{z}}{\frac{1}{z} - 1} = -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\sum_{n=1}^{\infty}\frac{1}{z^n}$$

Series expansion examples

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ for all } z$$

$$\begin{aligned} f(z) &= \frac{e^z}{z} \leftarrow \text{around } z = 0 \\ &= \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots \text{ for all } z \neq 0 \end{aligned}$$

$$\begin{aligned} f(z) &= e^{\frac{1}{z}} \leftarrow \text{around } z = \infty \\ &= \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots \text{ for all } z \neq 0 \end{aligned}$$

Example 6.5.2: Let us find the Laurent series of the function $e^z e^{1/z} = \sum_{n=-\infty}^{\infty} a_n z^n$.

- Show that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and $e^{1/z} = \sum_{m=0}^{\infty} \frac{1}{n! z^m}$.
- Show that $f(z)$ is analytic except for $z = 0$ and $z \to \infty$.
- Using $f(z) = f(1/z)$, show that $a_{-n} = a_n$.
- Show that a_0 is finite and $a_0 = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}$.
- Show that a_k is finite and $a_k = a_{-k} = \sum_{n=0}^{\infty} \frac{1}{n!(n+k)!}$.
- $e^z e^{1/z} = \sum_{n=0}^{\infty} \left[\frac{1}{(n!)^2} + \sum_{k=1}^{\infty} \frac{1}{n!(n+k)!} \left(z^k + \frac{1}{z^k} \right) \right]$.

6.6 Mapping Consider a complex function
$f(z) = w = u(x, y) + iv(x, y)$, $z = x + iy = re^{i\theta}$.
$z_0 = x_0 + iy_0 = r_0 e^{i\theta_0}$.

- Show that the transform $w = z + z_0$ translates any geometrical object in z-space by z_0.
- Show that under the transformation $w = z_0 z$ a circle of radius r in $z-$space is transformed into a circle with radius $|z_0|r$ and the phase is shifted by θ_0.
- Show that $w = \frac{1}{z} = \frac{1}{r} \cdot e^{i(-\theta)}$.
- Show that under the transformation $w = 1/z$ a disc of radius r in $z-$space is transformed into the outside of a disc with radius $1/r$.

Inversion: Consider the inversion

$w = u + iv = \frac{1}{z}, \;\; z = x + iy = re^{i\theta}.$

- Show that if $x^2 + y^2 = r^2$, then $u^2 + v^2 = (1/r)^2$.
- Using $u = x/r^2$ and $u^2 + v^2 = 1/r^2$, show that a vertical lint $x = x_0$ transforms into a cicle

$$\left(u - \frac{1}{2x_0}\right)^2 + v^2 = \left(\frac{1}{2x_0}\right)^2.$$

- Using $v = -y/r^2$ and $u^2 + v^2 = 1/r^2$, show that a horizontal line $y = y_0$ transforms into a cicle

$$u^2 + \left(v + \frac{1}{2y_0}\right)^2 = \left(\frac{1}{2y_0}\right)^2.$$

Using $z = re^{i\theta}$, show the following

- Show that a circle $|z| = r$ transforms into a ellipsis under $w = u + iv = z \pm \frac{1}{z}$:

$$u + iv = \left(r \pm \frac{1}{r}\right) \cos\theta + i \left(r \mp \frac{1}{r}\right) \sin\theta,$$

$$\frac{u^2}{\left(r \pm \frac{1}{r}\right)^2} + \frac{v^2}{\left(r \mp \frac{1}{r}\right)^2} = 1.$$

- Show that into limit $|z| \to 1$, $w = z + \frac{1}{z} \to u + i0$, where $-2 < u < 2$.

$\boldsymbol{f(z) = z^2 \,:\, 2 \longrightarrow 1}$ Show the following properties of the transformation $f(z) = z^2$.

$$
\begin{aligned}
z &= x + iy = re^{i\theta} \\
w &= \rho e^{i\phi} = z^2 = r^2 e^{i(2\theta)} \\
& \quad 0 < \theta < \pi \rightarrow 0 < \phi < 2\pi \\
& \quad \pi < \theta < 2\pi \rightarrow 2\pi < \phi < 4\pi \\
z_0^2 &= w \rightarrow (z_0 e^{i\pi})^2 = w, \text{ too}
\end{aligned}
$$

$$
\begin{aligned}
z^2 &= (x + iy)^2 = (x^2 - y^2) + i(2xy) \\
u &= x^2 - y^2 \\
v &= 2xy
\end{aligned}
$$

$\boldsymbol{f(z) = \sqrt{z} : 1 \rightarrow 2}$ Show that there are two roots of $\sqrt{z}$ for a single z:

$$
\begin{aligned}
z &= x + iy = re^{i(\theta+2k\pi)}, \ k = 0, 1, 2, \cdots \\
w &= \rho e^{i\phi} = z^{1/2} = \sqrt{r}e^{i(\theta+2k\pi)/2} \\
\phi &= \frac{\theta}{2}, \ \frac{\theta}{2} + \pi \\
\text{if} \quad & 0 \leq \theta < 2\pi \rightarrow \text{single} - \text{valued}
\end{aligned}
$$

Therefore, the function is multivalued unless we impose a branch cut.

$\boldsymbol{f(z) = e^z : \infty \to 1}$

$$
\begin{aligned}
z &= x + iy = re^{i(\theta+2k\pi)}, \;\; k = 0, 1, 2, \cdots \\
w &= e^{x+iy} = e^x e^{iy} \\
f(z + i2n\pi) &= f(z), \;\; n = \pm 1, \pm 2, \cdots \\
& \text{periodic}
\end{aligned}
$$

$\boldsymbol{f(z) = \ln z : 1 \to \infty}$ Show that there are infinitely many values of $\ln z$ for a single z:

$$\begin{aligned} z &= x + iy = re^{i(\theta+2k\pi)}, \;\; k = 0, 1, 2, \cdots \\ w &= \ln\left(re^{i(\theta+2n\pi)}\right) = \ln r + i2n\pi, \;\; n = 0, 1, 2, \cdots \\ \text{need} & \quad \text{cut}; -\pi < \theta \leq +\pi \\ &\to \text{single} - \text{valued} \end{aligned}$$

Therefore, the function is multivalued unless we impose a branch cut.

Conformal mapping: Let us consider the mapping $w = z^2$.

- Show that $u = a$ and $v = b$, where a, b are real constants, transforms into $x^2 + y^2 = a$ and $2xy = b$.
- Show that $u = a$ and $v = b$ are orthogonal.
- Show that the normal vector to $x^2 + y^2 = a$ is $(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (2x, 2y)$.
- Show that the normal vector to $2xy = b$ is $(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}) = (2y, 2x)$.
- Show that the two tangent at a common point $z = x + iy$ are orthogonal.

We will see any pair of orthogonal curves are mapped into orthogonal curves if the mapping function is analytic.

Let us consider the mapping $w = f(z) = u(x, y) + iv(x, y)$. Choose two curves $u(x, y) = a$ and $v(x, y) = b$ passing (x, y) in the z-plain, where a and b are real constants.

- Show that the normal vector to $u(x, y) = a$ is $(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$.
- Show that the normal vector to $v(x, y) = b$ is $(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y})$.
- Show that the inner product of the two 2-dimensional normal vectors vanishes if $f(z)$ is analytic.

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0,$$

due to the Cauchy-Riemann condition of analyticity.

Consider an analytic function $w = f(z)$. We will verify that the mapping preserves the angle.

- Show that $df(z)/dz$ exists and unique at a point $z = z_0$.
- Show that $\arg\left(\frac{df(z)}{dz}\right) = \alpha$, where α is real and constant at $z = z_0$. Is $df(z)/dz$ independent of the path approaching $z = z_0$?
- Show that $\arg[df(z)]$ =$\arg(dz)$+$\arg(\alpha)$.
- Choose two paths approaching $z = z_0$, $z_0 + \epsilon e^{i\theta_1}$ and $z_0 + \epsilon e^{i\theta_2}$, where θ_1 and θ_2 are constants and we vary $\epsilon \longrightarrow 0$. dz for the two paths are $e^{i\theta_1} d\epsilon$ and $e^{i\theta_2} d\epsilon$. The relative angle between the two paths are $\theta_1 - \theta_2$. Show that the corresponding path in the $w-$plain is $df[z_0 + \epsilon e^{i\theta_1}] - df[z_0 + \epsilon e^{i\theta_2}] = \theta_1 - \theta_2$.

Consider a two semi-infinite plates crossing with an angle θ_0 at the ends of their plains. Choose the cylindrical coordinate system where the edge is placed at the origin and the $x-$axis is placed on the plain and is normal to the edge.

- Show that the sector $0 < \theta < \theta_0$ is transformed into a strip in the w-plain.
- Assume the electric potential is $V(\theta = 0) = 0$ and $V(\theta = \theta_0) = V_0$.
- Use the symmetry to show that the potential at angle θ is $V(\theta) = V_0\theta/\theta_0 = \frac{V_0}{\theta_0}\,\mathrm{Im}(\ln z)$.
- Show that $w = U + iV = \frac{V_0}{\theta_0} \ln z$ is analytic and $\mathrm{Im}(w) = V$.
- Show that $E_x = -\frac{\partial V}{\partial x}$ and $E_y = -\frac{\partial V}{\partial y}$.

- Show that $\frac{dw}{dz} = -i(E_x - iE_y)$ and therefore $E_x = \text{Im}\left(-\frac{dw}{dz}\right)$ and $E_y = \text{Re}\left(-\frac{dw}{dz}\right)$.
- Differentiating the complex potential, find the electric field components

$$\begin{aligned}
-\frac{dw}{dz} &= -\frac{V_0}{z\theta_0} = \frac{V_0}{\theta_0}\left(-\frac{x}{r^2} + i\frac{y}{r^2}\right) \\
E_x &= \frac{V_0}{z\theta_0}\frac{y}{r^2} = \frac{V_0}{\theta_0}\frac{\sin\theta}{r} \\
E_y &= -\frac{V_0}{z\theta_0}\frac{x}{r^2} = -\frac{V_0}{\theta_0}\frac{\cos\theta}{r}
\end{aligned}$$

Homework set 4: (due: Oct. 16, 2004)

1. (6.5.3) Funtion $f(z)$ is analytic on and within the unit circle C. Also, $|f(z)| < 1$ for $|z| < 1$ and $f(0) = 0$. Show that $|f(z)| < |z|$ for $|z| \leq 1$.

2. Show that the Laurent series $e^z e^{1/z} = \sum_{n=0}^{\infty} \left[\frac{1}{(n!)^2} + \sum_{k=1}^{\infty} \frac{1}{n!(n+k)!} \left(z^k + \frac{1}{z^k}\right)\right]$ is convergent $\forall z \neq 0$.

3. (6.5.8) Show that the Laurent expansion of $f(z) = (e^z - 1)^{-1}$ about the origin is

$$\begin{aligned} f(z) &= \frac{1}{z}\left(\frac{z}{e^z - 1}\right) = \frac{1}{z}\left(1 + \frac{z}{2} + \frac{z^2}{6} + \cdots\right)^{-1} \\ &= \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \cdots \end{aligned}$$

4. (6.5.11)
(a) Given $f_1(z) = \int_0^{\infty} e^{-zt} dt$(with real t), show that the domain in which $f_1(z)$ exists and it analytic is $\text{Re}(z) > 0$.
(b) Show that $f_2(z) = 1/z$ equals $f_1(z)$ over $\text{Re}(z) > 0$ and is therefore an analytic continuation of $f_1(z)$ over the entire z-plain except for $z = 0$.
(c) Expand $1/z = 1/[i + (z - i)]$ about the point $z = i$ to find $1/z = -i \sum_{n=0}^{\infty} i^n (z - i)^n$ for $|z - i| < 1$.

5. (6.6.2) a) Show that the mapping $w = \frac{z-1}{z+1}$ transforms the right half of the z-plain($\text{Re}(z) > 0$) into the unit disc $|w| < 1$.
b) Show that the mapping $w = \frac{z-i}{z+i}$ transforms the upper half of the z-plain($\text{Im}(z) < 0$) into the unit disc $|w| < 1$.

Mid-term Exam:

Chapter 4 and 6

Oct. 18, 2004, Monday

Chapter 7
Complex Variable II

- We now know many properties of analytic functions.
- We extensively use Cauchy integral theorem to evaluate many important definite integrals.
- **entire function**: Functions such as z and e^z are analytic everywhere.
- **singularity**: Function such as $1/z$ has sigularity at $z = 0$. The function is not analytic at the singular point. The point is isolated because anywhere near the point the function is analytic.
- **meromorphic function**: a function is meromorphic if it has a finite number of singular points.

Poles: A series expansion near an isolated pole can be done using Laurent series method. Consider a Laurent series exapnsion about z_0.

$$
\begin{aligned}
f(z) &= \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \\
&= a_0 + \sum_{n=1}^{\infty} \left[a_n (z - z_0)^n + \frac{a_{-n}}{(z - z_0)^n} \right] \\
\frac{a_{-n}}{(z - z_0)^n} &= \text{pole of order } n \\
a_{-1} &= \text{Residue}
\end{aligned}
$$

Essential singularity: If a series has a pole of infinite order, the function has essential singularity at the point. $e^{1/z}$ as essential singularity at $z = 0$. Laurent series exapnsion about $|z| = \infty$.

$$
\begin{aligned}
e^{1/z} &= \sum_{k=0}^{\infty} \frac{1}{n! z^n} \text{ poles at } z = 0 \text{ for all } n \\
a_{-n} &= \frac{1}{n!}, \text{ pole of any order } n = 1, 2, \cdots \rightarrow \text{essential singularity}
\end{aligned}
$$

The real function $\sin x$ is bounded. However, $\sin z$ also has an essential singularity at $z \to \infty (\frac{1}{z} = t \to 0)$;

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! t^{2n+1}}.$$

Show that $\sin z = \sin x \cosh y + i \cos x \sinh y$ and that $\sin z$ is not bounded as $\mathrm{Im}(z) \to \pm\infty$.

Branch cut

- Show that $\ln z = \ln r + i\theta$ is single valued only if we impose a branch cut.
- Show that the cut of $\mathrm{Re} x < 0$ and $y = 0$ is a choice and the answer has the same limiting value as z approaches the positive real axis. ($\mathrm{Re} x > 0$ and $y = 0$)
- Show that $z = e^{\ln z}$ and $\ln e^z$ is not always equal to z.
- Show that $z^a = r^a e^{ia\theta}$ is multivalued unless we impose a cut.

$$e^{ia2\pi} \neq e^{i0}$$

unless $a =$ integer. The cut must pass the branch point $z = 0$.

Functions with 2 branch points

$$
\begin{aligned}
(z^2-1)^{1/2} &= (z+1)^{1/2}(z-1)^{1/2} \\
\text{if} &\quad z = x, \ -1 < x < 1 \\
(z^2-1)^{1/2} &= i\sqrt{1-x^2}, \ -i\sqrt{1-x^2} \ : \ \text{double} \\
&\rightarrow \ \text{branch points are } z = \pm 1 \\
&\rightarrow \ \text{need a common branch cut} \\
&\quad z = x, \ -1 < x < 1 \\
(z^2-1)^{1/2} &= \sqrt{r_+ r_-} e^{\frac{i}{2}(\theta_+ + \theta_-)} \\
z - 1 &= r_+ e^{i\theta_+}, \ -\pi < \theta_+ < \pi \\
z + 1 &= r_- e^{i\theta_-}, \ 0 < \theta_- < 2\pi \\
&\rightarrow \ -\frac{\pi}{2} < \frac{1}{2}(\theta_+ + \theta_-) < \frac{3\pi}{2} \rightarrow \text{single} - \text{valued}
\end{aligned}
$$

Uniqueness Theorem for power series(Sec. 5.7): Assume there are two series expansions of a function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad -R_a < x < R_a$$
$$= \sum_{n=0}^{\infty} b_n x^n, \qquad -R_b < x < R_b$$

with overlapping intervals of convergence, including the origin.

- Substituting $x = 0$, show that $a_0 = b_0$.
- Differentiating both sides once and substituting $x = 0$, show that $a_1 = b_1$. Using mathematical induction, show that $a_n = b_n$ for all n. Therefore, Talyor expansion is unique.

Consider a function $f(z)$ having an order-n pole at $z = z_0$. One can expand $f(z)$ around $z = z_0$ in terms of Laurent series expansion. $f(z) = \sum_{k=-n}^{\infty} a_k z^k, \quad a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{k+1}}$, where the contour C is enclosing z_0.

- Show that $(z - z_0)^n f(z) = (z - z_0)^{n-1} [a_{-1} + o(z - z_0)]$, where $o(0) = 0$.
- Show that $\frac{d^{n-1}}{dz^{n-1}}(z - z_0)^{n-1} = (n - 1)!$ and $\left[\frac{d^{n-1}}{do(z-z_0)}\right]_{z=z_0} = 0$.
- Show that the residue a_{-1} is then

$$a_{-1} = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z)\right]_{z=z_0}$$

Residue Theorem: Assume $f(z)$ has poles only at $z = z_0$.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad a_{-1} = \text{residue}$$

$$\oint_C f(z) dz = \begin{cases} 2\pi i a_{-1}, \ z_0 \ \text{ is inside } C \\ 0, \ z_0 \ \text{ is outside } C \end{cases}$$

Assume $f(z)$ has poles at $z = z_1,\ z_2, \cdots$ as

$$f(z) = \sum_{n=-\infty}^{\infty} \left[(a_1)_n (z - z_1)^n + (a_k)_n (z - z_2)^n + \cdots\right].$$

- Show that $\sum_{n=-\infty}^{\infty} a_n (z - z_i)^n$ is analytic for all $z_j \neq z_i$.
- Show if a contour encloses poles z_1 through z_m, show that

$$\oint_C f(z) dz = 2\pi i \left[(a_1)_{-1} + (a_2)_{-1} + \cdots + (a_k)_{-1}\right].$$

Calulate the residues

$$f(z) = \frac{1}{z-i} \rightarrow \text{Residue}(z=i) = 1$$

$$f(z) = \frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)}$$

$$a_{-1}(z=1) = \frac{1}{(1+1)} = \frac{1}{2}, \quad a_{-1}(z=-1) = \frac{1}{(-1-1)} = -\frac{1}{2}$$

Find the residue of the following function at $z = 0$.

$$\begin{aligned} f(z) &= \frac{1}{z^2(z-1)} \rightarrow -\frac{1}{z^2}(1+z+z^2+\cdots) = -\frac{1}{z^2} - \frac{1}{z} - \cdots \\ \text{Res}(0) &= \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]_{z=z_0} \leftarrow n=2,\ z_0=0 \\ &= \left[-\frac{1}{(z-1)^2}\right]_{z=1} = -1 \end{aligned}$$

Example 7.2.1: Let us evaluate the definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + \epsilon \cos\theta}, \;\; |\epsilon| < 1$$

- Using the following change of varibale,

$$\begin{aligned} z &= e^{i\theta}, \; dz = ie^{i\theta} d\theta \rightarrow d\theta = -i\frac{dz}{z} \\ 1 + \epsilon\cos\theta &= 1 + \epsilon\frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{\epsilon}{2z}\left(z^2 + \frac{2z}{\epsilon} + 1\right) \end{aligned}$$

  show that the integral can be expressed as a contour integral over a unit circle as

$$
\begin{aligned}
I &= \int_0^{2\pi} \frac{d\theta}{1+\epsilon\cos\theta} = \oint_C \left(-i\frac{dz}{z}\right)\frac{2z}{\epsilon(z-z_+)(z-z_-)} \\
&= \frac{-2i}{\epsilon}\oint_C \frac{dz}{(z-z_+)(z-z_-)} \\
z_\pm &= -\frac{1}{\epsilon}\left(1\pm\sqrt{1-\epsilon^2}\right), \quad z_+ - z_- = \frac{2}{\epsilon}\sqrt{1-\epsilon^2}
\end{aligned}
$$

- Show that $|z_+| < 1$ and $z_- < -1$; only z_+ is enclosed by the contour of the unit circle C.

- Show that the residue for $[(z-z_-)(z-z_+)]^{-1}$ at $z = z_+$ is

$$
\frac{1}{z_+ - z_-} = \frac{\epsilon}{2\sqrt{1-\epsilon^2}}.
$$

- Finally

$$
\begin{aligned}
I &= \int_0^{2\pi} \frac{d\theta}{1+\epsilon\cos\theta}, \ \ |\epsilon| < 1 \\
&= \frac{-2i}{\epsilon}\frac{2\pi i}{z_+ - z_-} = \frac{-2i}{\epsilon}2\pi i\frac{\epsilon}{2\sqrt{1-\epsilon^2}} = \frac{2\pi}{\sqrt{1-\epsilon^2}}
\end{aligned}
$$

Example 7.2.2: Let us evaluate the definite integral of a real variable

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

using the complex contour integral technique.

- Show that $I = \lim_{R\to\infty} \int_{-R}^{R} \frac{dz}{1+z^2}$, where $z = x + i0$.
- Let us take a contour C made of C_1, from $-R + i0$ to $R + i0$, and C_2 along $Re^{i\theta}$, where $0 < \theta < \pi$. Show that

$$J = \int_C \frac{dz}{z^2+1} = \int_C \frac{dz}{(z+i)(z-i)} = I + \int_{C_2} \frac{dz}{z^2+1}.$$

- Show that $z = i$ is the only pole enclosed by C and its residue is

$$a_{-1}(z=i) = \frac{1}{2i} \to J = 2\pi i \times a_{-1}(z=i) = \pi.$$

- Show that the integral along the large semi-circle C_2 is reduced into

$$\int_{C_2} \frac{dz}{z^2+1} = \int_{\theta=0}^{2\pi} \frac{d(Re^{i\theta})}{1+(Re^{i\theta})^2} = Rie^{i\theta} \int_0^{2\pi} \frac{d\theta}{1+R^2e^{2i\theta}}$$

- Using $|\int f(z)dz| \leq |f_{\max}|L$, where $|f_{\max}|$ is the maxmum value of the $|f(z)|$ along the path and L is the length of the path, show that

$$\int_{C_2} \frac{dz}{z^2+1} \leq \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.$$

- Therefore

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi, \quad \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Example 7.2.4: Let us evaluate the definite integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \text{Im} \int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz.$$

- Take the contour $C = [-R + i0 \rightarrow -\delta + i0]$ $+[C_1 : \delta e^{i\theta}, \theta : \pi \rightarrow 0]$ $+[-R + i0 \rightarrow -\delta + i0]$ $+[C_2 : Re^{i\theta}, \theta : 0 \rightarrow \pi]$.

- Show that the function e^{iz}/z is analytic in the region enclosed by the contour C. Therefore,

$$\oint_C \frac{e^{iz}}{z} dz = 0.$$

- Show that in the limit $\delta \rightarrow 0$ the integral over the semi-circle C_1 becomes

$$\int_{C_1} \frac{e^{iz}}{z} dz = (\pi i)_{\text{half circle}} (-1)_{\text{clockwise}} = -\pi i$$

note that the path is in the negative sense.

- Show that in the limit $R \rightarrow \infty$ the integral over the large semi-circle C_2 vanishes

$$\begin{aligned}
\left| \int_{C_2} \frac{e^{iz}}{z} dz \right| &\leq \left| i \int_0^{\pi} e^{iR\cos\theta - R\sin\theta} d\theta \right| \leftarrow z = Re^{i\theta},\ \frac{dz}{z} = i d\theta \\
&= \int_0^{\pi} e^{-R\sin\theta} d\theta = 2 \int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta \\
&\leq 2 \int_0^{\frac{\pi}{2}} e^{-R\frac{2\theta}{\pi}} d\theta = \frac{\pi}{R} \left(1 - e^{-R}\right) \rightarrow 0 \text{ as } R \rightarrow 0.
\end{aligned}$$

- Show that

$$\oint_C \frac{e^{iz}}{z} = i\pi = \int_{-R}^{-\delta} \frac{e^{ix}}{x} + \int_{\delta}^{R} \frac{e^{ix}}{x}$$

- Show that

$$\begin{aligned}
\lim_{\delta\to 0} \int_{-\delta}^{\delta} \frac{\sin x}{x} dx &= \lim_{\delta\to 0} \int_{-\delta}^{\delta} \frac{x + o(x^3)}{x} dx \\
&= \lim_{\delta\to 0} \int_{-\delta}^{\delta} \left(1 + o(x^2) + \cdots\right) dx \\
&\to \lim_{\delta\to 0} \left[2\delta + o(\delta^3)\right] \to 0
\end{aligned}$$

- Show that

$$\oint_C \frac{\cos z}{z} = 0, \quad \oint_C \frac{\sin z}{z} = \pi$$

$$\int_0^{\infty} \frac{\sin z}{z} = \int_{-\infty}^{0} \frac{\sin z}{z} = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{\sin z}{z} = \pi$$

(7.2.11): Let us use the same method to calculate the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$$

- Show that the integral can be reparametrized as

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx &= \int_{-\infty}^{\infty} \frac{1-\cos 2x}{2x^2} dx = \text{Re} \int_{-\infty}^{\infty} \frac{1-e^{i2z}}{2z^2} dz \\
\text{Residue}(0) &= -\frac{2i}{2} = -i \\
\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx &= \pi \quad \int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.
\end{aligned}$$

This integral appears when we derive Fermi's Golden Rule. See time-dependent perturbation theory in quantum mechanics.

Feynman propagator$(\epsilon \to 0^+,\ \omega_0 > 0)$

$$\begin{aligned}
i\Delta_F(t) &= i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega^2-\omega_0^2+i\epsilon} \\
&= i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{(\omega-\omega_0+i\epsilon)(\omega+\omega_0-i\epsilon)} \\
\text{closing } C &: \quad t>0 \to \text{clockw.},\ t<0 \to \text{counterclockw.} \\
\mathrm{Res}(\omega_0) &= \frac{e^{-i\omega_0 t}}{2\omega_0},\ \mathrm{Res}(-\omega_0) = \frac{e^{+i\omega_0 t}}{-2\omega_0} \\
i\Delta_F(t) &= \frac{i(2\pi i)}{2\pi}\left[(-1)_{\text{c.clockw.}}\frac{\theta(t)e^{-i\omega_0 t}}{2\omega_0} + (+1)_{\text{clockw.}}\frac{\theta(-t)e^{i\omega_0 t}}{-2\omega_0}\right] \\
&= \frac{1}{2\omega_0}\left[\theta(t)e^{-i\omega_0 t} + \theta(-t)e^{i\omega_0 t}\right]
\end{aligned}$$

Mittag-Leffler Theorem: $f(z)$ has poles $z_1, \cdots, z_n$ inside C_n(center 0). $|f(z_n)|/R_n \to 0$ as $R_n \to \infty$ (bounded). $z \neq z_i, 0, C_n$. $\text{Res}(z_i) = b_i$.

$$\begin{aligned}
I_n(z) &= \frac{1}{2\pi i}\int_{C_n} \frac{f(w)}{w(w-z)} dw \leftarrow \text{poles} : z_i,\ 0,\ w \\
&= \sum \text{Res} = \sum_{m=1}^{n} \frac{b_m}{z_m(z_m - z)} + \frac{f(z) - f(0)}{z} \\
|I_n| &\leq \frac{2\pi R_n}{2\pi} \frac{|f(w)|_{\max}}{R_n(R_n - |z|)} \to 0,\ \text{as } R_n \to \infty\ (R_n \gg |z|) \\
f(z) - f(0) &= \sum_{m=1}^{\infty} \frac{z b_m}{z_m(z - z_m)} = \sum_{m=1}^{\infty} b_m \left[\frac{1}{z - z_m} + \frac{1}{z_m}\right]
\end{aligned}$$

Example 7.2.7 Mittag-Leffler Theorem application:

$f(z) = \pi \cot \pi z - \frac{1}{z}$ has poles $z = n, \; n = \pm 1, \pm 2 \cdots$.

$$
\begin{aligned}
f(0) &= \lim_{z \to 0} \left(\frac{\pi \cos \pi z}{\sin \pi z} - \frac{1}{z} \right) = 0 \\
b_n &= \mathrm{Res}(n) = \left[\frac{\pi \cos \pi z}{(\sin \pi z)'} \right]_{z=n} = \frac{\pi \cos n\pi}{\pi \cos n\pi} = 1 \\
f(z) &= \sum_{n=1}^{\infty} \left[\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z-(-n)} + \frac{1}{(-n)} \right] \\
&= \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}
\end{aligned}
$$

Weierstrass' Factorization Formula : $\frac{f'(z)}{f(z)}$, $f(z) = (z - z_n)g(z)$, $g(z)$ is analytic and $g(z_n) \neq 0$. If Mittag-Leffler Theorem is applicable to $\frac{f'(z)}{f(z)}$,

$$\begin{aligned}
\frac{f'(z)}{f(z)} &= \frac{[(z-z_n)g(z)]'}{(z-z_n)g(z)} = \frac{1}{z-z_n} + \frac{g'(z)}{g(z)} \\
\frac{f'(z)}{f(z)} &= \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\frac{1}{z-z_n} + \frac{1}{z_n}\right] \ \text{(Mittag − Leffler)} \\
\ln\frac{f(z)}{f(0)} &= \ln f(z) - \ln f(0) = \int_{f(0)}^{f(z)} \frac{df(w)}{f(w)} = \int_0^z \frac{f'(w)}{f(w)} dw \\
&= \frac{zf'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\ln\left(\frac{z-z_n}{-z_n}\right) + \frac{z}{z_n}\right] \\
f(z) &= f(0)e^{\frac{zf'(0)}{f(0)}}\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}}
\end{aligned}$$

Weierstrass' Factorization Formula Application:

$$f(z) = f(0)e^{\frac{zf'(0)}{f(0)}} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}}$$

$$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3} + \cdots, \; f(0) = 1, \; f'(0) = 0$$

$$\frac{\sin z}{z} = \prod_{n \neq 0, n=-\infty}^{\infty} \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}} = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right)\left(1 + \frac{z}{n\pi}\right) e^{\frac{z}{n\pi} - \frac{z}{n\pi}}$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

Weierstrass' Factorization Formula Application:

$$f(z) = f(0)e^{\frac{zf'(0)}{f(0)}} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}}$$

$$f(z) = \cos z = 1 - \frac{z^2}{2} + \cdots\ ,\ f(0) = 1,\ f'(0) = 0$$

$$\cos z = \prod_{n=1}^{\infty} \left[1 - \frac{z}{\left(n - \frac{1}{2}\right)\pi}\right] e^{\frac{z}{\left(n-\frac{1}{2}\right)\pi}} \left[1 + \frac{z}{\left(n - \frac{1}{2}\right)\pi}\right] e^{-\frac{z}{\left(n-\frac{1}{2}\right)\pi}}$$

$$= \prod_{n=1}^{\infty} \left[1 - \frac{z^2}{\left(n - \frac{1}{2}\right)^2 \pi^2}\right]$$

Example 7.2.5) Bessel funtion

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

$$J_n(x) = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^{n+1}} dt$$

$$\rightarrow \text{ Laurent coefficient}$$

$$C = re^{i\theta}, \text{ for any integer } n$$

Homework set 4: (due: Oct. 16, 2004)

1. (7.1.2) There is a function of the form

$$f(z) = \frac{f_1(z)}{f_2(z)},$$

   where $f_i(z)$'s are analytic, $f_2(z_0) = 0$, $f_1(z_0) \neq 0$, and $f_2'(z_0) \neq 0$. Show that $f(z)$ has a pole of order 1 at $z = z_0$. Show that the residue a_{-1} for the function at $z = z_0$ is

$$a_{-1} = \frac{f_1(z_0)}{f_2'(z_0)}.$$

2. Using above result show that $a_{-1} = -\frac{i}{2}$ at $z = i$ if $f(z) = \frac{1}{z^2+1}$.