Quantum Electrodynamics

$$Energy - momentum\ relation\ in\ special\ relativity$$

$$\frac{E^2}{c^2} - \vec{p}^2 = (mc)^2$$

$$(\frac{E}{c} - \vec{\sigma} \circ \vec{p})(\frac{E}{c} + \vec{\sigma} \circ \vec{p}) = (mc)^2$$

$$\vec{\sigma} \Rightarrow Pauli's\ matrices$$

$$(\vec{\sigma} \circ \vec{a})(\vec{\sigma} \circ \vec{b}) = \vec{a} \circ \vec{b} + i\vec{\sigma}(\vec{a} \times \vec{b})$$

$$\therefore (\vec{\sigma} \circ \vec{p})(\vec{\sigma} \circ \vec{p}) = \vec{p}^2$$

$Spin - \frac{1}{2}$ 입자에 적용할 수 있다.

$$\frac{E}{c} \to i\hbar \frac{\partial}{\partial t} = i\hbar \frac{\partial}{\partial x_0}$$

$$\vec{p} = -i\hbar \nabla$$

위와같이 연산자를 바꾸면 된다.

$$(i\hbar \frac{\partial}{\partial x_0} + \vec{\sigma} \circ i\hbar \nabla)(i\hbar \frac{\partial}{\partial x_0} - \vec{\sigma} \circ i\hbar \nabla)\Psi = (mc)^2 \Psi$$

$$\Psi^{(L)}(x)=\Psi(x)$$

$$\Psi^{(R)}(x)=\frac{1}{mc}(i\hbar\frac{\partial}{\partial x_0}-i\hbar\vec{\sigma}\circ\nabla)\Psi(x)$$로 정의하면

$$(i\hbar\vec{\sigma}\circ\nabla-i\hbar\frac{\partial}{\partial x_0})\Psi^{(L)}(x)=-mc\Psi^{(R)}(x)$$

$$(i\hbar\vec{\sigma}\circ\nabla+i\hbar\frac{\partial}{\partial x_0})\Psi^{(R)}(x)=mc\Psi^{(L)}(x)$$으로 연관되어 있다.

위 두 식의 차와 합에 의해서

$$i\hbar\vec{\sigma}\circ\nabla[\Psi^{(R)}(x)-\Psi^{(L)}(x)]-i\hbar\frac{\partial}{\partial x_0}[\Psi^{(L)}(x)+\Psi^{(R)}(x)]=-mc[\Psi^{(L)}(x)+\Psi^{(R)}(x)]$$

$$i\hbar\vec{\sigma}\circ\nabla[\Psi^{(R)}(x)+\Psi^{(L)}(x)]+i\hbar\frac{\partial}{\partial x_0}[\Psi^{(R)}(x)-\Psi^{(L)}(x)]=-mc[\Psi^{(R)}(x)-\Psi^{(L)}(x)]$$

두 식을 행렬로 다시쓰면

$$\begin{pmatrix}-i\hbar\frac{\partial}{\partial x_0} & i\hbar\vec{\sigma}\circ\nabla\\ i\hbar\vec{\sigma}\circ\nabla & i\hbar\frac{\partial}{\partial x_0}\end{pmatrix}\begin{pmatrix}\Psi^{(R)}(x)+\Psi^{(L)}(x)\\ \Psi^{(R)}(x)-\Psi^{(L)}(x)\end{pmatrix}=-mc\begin{pmatrix}\Psi^{(R)}(x)+\Psi^{(L)}(x)\\ \Psi^{(R)}(x)-\Psi^{(L)}(x)\end{pmatrix}$$

$$\Psi(x)=\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}=\begin{pmatrix}\Psi^{(R)}(x)+\Psi^{(L)}(x)\\ \Psi^{(R)}(x)-\Psi^{(L)}(x)\end{pmatrix}$$

$$\begin{pmatrix}-i\hbar\frac{\partial}{\partial x_0} & i\hbar\vec{\sigma}\circ\nabla\\ i\hbar\vec{\sigma}\circ\nabla & i\hbar\frac{\partial}{\partial x_0}\end{pmatrix}\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}=-mc\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}$$

$\gamma-matrix$

$$\gamma^0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},\qquad \vec{\gamma}=\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}$$

$$(i\hbar\gamma^{\mu}\partial_{\mu}-mc)\Psi(x)=0$$

$$\Psi(x)=?$$

Dirac plane wave

$$(\gamma^{\mu}\frac{\partial}{\partial x_{\mu}}+\frac{mc}{\hbar})\Psi(x)=0, \qquad \Psi(x)=u(p)e^{i(p_{\mu}x_{\mu}-Et)/\hbar}$$

$$\Rightarrow (\gamma^{\mu}i\frac{p_{\mu}}{\hbar}+\frac{mc}{\hbar})\Psi(x)=0$$

행렬로 표현해보면,

$$[\frac{1}{\hbar}\begin{pmatrix} 0 & \vec{p}\circ\vec{\sigma} \\ -\vec{p}\circ\vec{\sigma} & 0 \end{pmatrix}+\frac{1}{\hbar}\begin{pmatrix} -\frac{E}{c} & 0 \\ 0 & \frac{E}{c} \end{pmatrix}+\frac{mc}{\hbar}]\Psi(x)=0$$

$$\Rightarrow \begin{pmatrix} -\frac{E}{c}+mc & \vec{p}\circ\vec{\sigma} \\ -\vec{p}\circ\vec{\sigma} & \frac{E}{c}+mc \end{pmatrix}\Psi(x)=0$$

$$\Rightarrow \begin{pmatrix} -E+mc^2 & 0 & p_3 & p_1-ip_2 \\ 0 & -E+mc^2 & p_1+ip_2 & -p_3 \\ p_3 & p_1-ip_2 & E+mc^2 & 0 \\ p_1+ip_2 & -p_3 & 0 & E+mc^2 \end{pmatrix}\Psi(x)=0,$$

$$\Psi_1=\begin{pmatrix} 1 \\ 0 \\ X_{11} \\ X_{12} \end{pmatrix}, \Psi_2=\begin{pmatrix} 0 \\ 1 \\ X_{21} \\ X_{22} \end{pmatrix}, \Psi_3=\begin{pmatrix} Y_{11} \\ Y_{12} \\ 1 \\ 0 \end{pmatrix}, \Psi_4=\begin{pmatrix} Y_{21} \\ Y_{22} \\ 0 \\ 1 \end{pmatrix}$$

$$\Psi^{+}\Psi = \frac{1}{V}(one\ \ particle\ \ per\ unit\ volume)$$

$$(A)\Psi_1 = \begin{pmatrix} -E + mc^2 + X_1 p_3 c + X_{12}(p_1 - ip_2)c \\ X_{11}(p_1 + ip_2)c - X_{12} p_3 c \\ -p_3 c + X_{11}(E + mc^2) \\ -(p_1 + ip_2)c + X_{12}(E + mc^2) \end{pmatrix} = 0, \Rightarrow \Psi_1 = \sqrt{\frac{|E| + mc^2}{2|E|V}} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3 c}{E + mc^2} \\ \frac{(p_1 + ip_2)c}{E + mc^2} \end{pmatrix}$$

$$(A)\Psi_2 = \begin{pmatrix} X_{21} p_3 c + X_{22}(p_1 - ip_2)c \\ -E + mc^2 + X_{21}(p_1 + ip_2)c - X_{22} p_3 c \\ -(p_1 + ip_2)c + X_{21}(E + mc^2) \\ p_3 c + X_{22}(E + mc^2) \end{pmatrix} = 0, \Rightarrow \Psi_2 = \sqrt{\frac{|E| + mc^2}{2|E|V}} \begin{pmatrix} 0 \\ 1 \\ \frac{(p_1 - ip_2)c}{E + mc^2} \\ \frac{-p_3 c}{E + mc^2} \end{pmatrix}$$

$$(A)\Psi_3 = \begin{pmatrix} Y_{11}(-E + mc^2) + p_3 c \\ Y_{12}(-E + mc^2) + (p_1 + ip_2)c \\ -Y_{11} p_3 c - Y_{12}(p_1 - ip_2) + (E + mc^2) \\ -Y_{11}(p_1 + ip_2)c + Y_{12} p_3 c \end{pmatrix} = 0, \Rightarrow \Psi_3 = \sqrt{\frac{|E| + mc^2}{2|E|V}} \begin{pmatrix} \frac{-p_3 c}{-E + mc^2} \\ \frac{-(p_1 + ip_2)c}{-E + mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$(A)\Psi_4 = \begin{pmatrix} Y_{21}(-E + mc^2) + (p_1 - ip_2)c \\ Y_{22}(-E + mc^2) - p_3 c \\ Y_{21}(-p_3 c) - Y_{22}(p_1 - ip_2)c \\ -Y_{21}(p_1 + ip_2)c + Y_{22} p_3 c + (E + mc^2) \end{pmatrix} = 0, \Rightarrow \Psi_4 = \sqrt{\frac{|E| + mc^2}{2|E|V}} \begin{pmatrix} \frac{-(p_1 - ip_2)c}{-E + mc^2} \\ \frac{p_3 c}{-E + mc^2} \\ 0 \\ 1 \end{pmatrix}$$