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Extended maximum likelihood method

KIHYEON CHO
DEPARTMENT OF PHYSICS AND ASTRONOMY
UNIVERSITY OF TENNESSEE, KNOXVILLE, TN 37996-1200

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1 Maximum likelihood method

The partial wave analysis is done by using the maximum likelihood method to determine the parameters for the productions amplitudes ${}^{\epsilon}V_b$ for a given spin-parity state b and reflectivity ϵ [1, 2].

Let us consider a sample of N predicted events in a bin of area Δm. The likelihood function may be written as a product of probabilities:

$$L_{likh} = \prod_i^N P^{(i)}.$$

We can denote the probability with intensity distributions:

$$L_{likh} = \prod_i^N I(\tau_i) \tag{1}$$

$$= \prod_i^N |\sum_{\epsilon,b} {}^{\epsilon}A_b(\tau_i)\ {}^{\epsilon}V_b|^2 \tag{2}$$

where ${}^{\epsilon}A_b(\tau_i)$ is decay amplitude for a given spin-parity state b and reflectivity ϵ. τ stands for all the variables relevant for partial wave analysis. At this point, we stress that it is essential to remember to include the normalization factor at maximum likelihood method [3]. Therefore, the Eq. 2 becomes

$$L_{likh} = \prod_i^N \frac{I(\tau_i)}{\int I(\tau)d\tau} \tag{3}$$

$$= \prod_i^N \frac{|\sum_{\epsilon,b} \ {}^{\epsilon}A_b(\tau_i) \ {}^{\epsilon}V_b|^2}{\int |\sum_{\epsilon,b} \ {}^{\epsilon}A_b(\tau) \ {}^{\epsilon}V_b|^2 d\tau}. \tag{4}$$

2 Maximum likelihood method with detector acceptance

The Eq. 4 is considered at the level of the predicted number of N, which means that the equation is considered without detector acceptance. Since we have finite detector acceptance we consider the detector acceptance.

Let us think about the detector acceptance from the predicted number of events N. In a bin of area Δm, the accepted function $\eta(\tau_i)$ denotes the mapping from the predicted event to the detected event for the i-th event as shown Fig. 1.

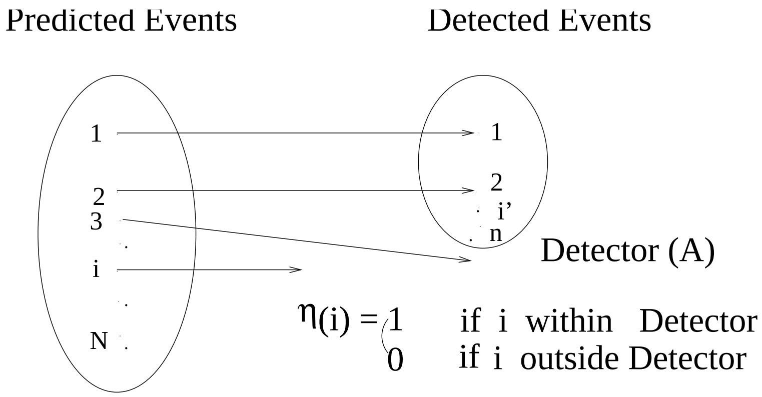

Figure 1: The mapping from the predicted event to the detected event by the accepted function $\eta(\tau_i)$.

As shown Fig. 1, the accepted function $\eta(\tau_i)$ has only two choices - either on or off at the detector.

$$\eta(\tau_i) = \begin{cases} 1, & \text{if ith event is within detector} \\ 0, & \text{if ith event is outside detector} \end{cases} \tag{5}$$

Now we apply detector acceptance for each predicted event. Therefore, the Eq. 3 becomes

$$L_{likh} = \prod_i^N \frac{I(\tau_i)\eta(\tau_i)}{\int I(\tau)\eta(\tau)d\tau}. \tag{6}$$

The i-th event at the each predicted events remains only detector accepted events after being applied by detector acceptance. Then, we collect only detector accepted events. Let us denote the total number of the detector accepted events as n. Then, it gives the detector efficiency ϵ which is $\epsilon = n/N$. We denote the index for the detector accepted event as i'. Therefore, we can rewrite the Eq. 6 as

$$L_{likh} = \prod_{i'}^{n} \frac{I(\tau_{i'})}{\int I(\tau)\eta(\tau)d\tau} \quad (7)$$

$$= \prod_{i}^{n} \frac{I(\tau_{i})}{\int I(\tau)\eta(\tau)d\tau} \quad (8)$$

At the last equation, we can replace i' with i. Please note that the number of events is changed from predicted number of events N to the observed number of events n after being applied by detector acceptance. Therefore, the maximum likelihood method after considering detector acceptance is

$$L_{likh} = \prod_{i}^{n} \frac{|\sum_{\epsilon,b} {}^{\epsilon}A_b(\tau_i)\ {}^{\epsilon}V_b|^2}{\int |\sum_{\epsilon,b} {}^{\epsilon}A_b(\tau)\ {}^{\epsilon}V_b|^2 \eta(\tau)d\tau}. \quad (9)$$

3 The extended maximum likelihood method

In the usual likelihood approach we determine parameters relevant to the shapes of distributions, but simply take the absolute normalizations as being equal to the observed number of events. However, at the partial wave analysis we use the extended maximum likelihood method [1].

Whether we should use the "maximum likelihood" or the "extended maximum likelihood" approach depends on the question we are trying to answer. If, for example, the tracks are the decaying products of some resonance, and we want to determine its branching ratios, then "maximum likelihood" is required. If, however, we are determining its partial decay rates, then we need the "extended maximum likelihood method" [4].

The extended likelihood function L includes an extra factor for the probability of obtaining a sample of size n from a Poisson distribution of $\bar{n}$. The meaning of $\bar{n}$ is the average number of events would expect if the same experiment were repeated many times. Therefore, the relation between extended maximum likelihood L and maximum likelihood L_{likh} is

$$L = [\frac{\bar{n}^n}{n!} e^{-\bar{n}}] L_{likh}.$$

Then, the extended likelihood function for finding n events of a given bin with an experimental acceptance $\eta(\tau)$ can be written as from Eq. 8

$$L = [\frac{\bar{n}^n}{n!} e^{-\bar{n}}] \prod_{i}^{n} [\frac{I(\tau_i)}{\int I(\tau)\eta(\tau)d\tau}]$$

where the first bracket is the Poisson probability for n events. This is the so called extended likelihood function, in the sense that the Poisson distribution for n itself is included in the likelihood function, i.e. it is one of the parameters to be determined. The quantity

$$\bar{n} = \int I(\tau)\eta(\tau)d\tau$$

is the number of events expected in the bin as predicted by the intensity function. The extended likelihood function can then be written as

$$L = \prod_i^n [I(\tau_i)] \exp[-\int I(\tau)\eta(\tau)d\tau]$$

where terms that do not depend on the parameters of the fit have been dropped. Therefore, the log of the extended likelihood function now has the form,

$$\ln L = \sum_i^n \ln I(\tau_i) - \int I(\tau)\eta(\tau)d\tau. \tag{10}$$

The integral in Eq. 10 can be calculated by the Monte Carlo method. We begin by inserting the expression for the distribution function $I(\tau)$

$$I(\tau) = |\sum_{\epsilon,b} {}^{\epsilon}A_b(\tau)\ {}^{\epsilon}V_b|^2 \tag{11}$$

into the integral. Now, because the fitted parameters do not depend on the kinematic variables τ, the order of the summation and integration can be exchanged. Therefore, $\bar{n}$ becomes

$$\begin{aligned} \bar{n} &= \int |\sum_{\epsilon,b} {}^{\epsilon}A_b(\tau)\ {}^{\epsilon}V_b|^2 \eta(\tau)d\tau & (12) \\ &= \sum_{\epsilon,b,b'} {}^{\epsilon}V_b\ {}^{\epsilon}V_{b'}^* \int {}^{\epsilon}A_b(\tau)\ {}^{\epsilon}A_{b'}^*(\tau)\eta(\tau)d\tau. & (13) \end{aligned}$$

The integral over the decay amplitudes are often called the accepted integrals [5]. Since the acceptance $\eta(\tau)$ is not known analytically, the accepted integrals are evaluated using Monte Carlo events. Therefore, the next step is to approximate the acceptance as sum over accepted Monte Carlo events divided by the number of generated events [6]. Let us define M as the total number of Monte Carlo events generated and M_x as the total number of accepted Monte Carlo events by the finite geometry of the experiment and other analysis cuts. Then, we need two sets of normalizations. One is

$${}^{\epsilon}\Psi_{b,b'} = \frac{1}{M}\sum_{i=1}^{M} {}^{\epsilon}A_b(\tau_i)\ {}^{\epsilon}A_{b'}^*(\tau_i)$$

for the raw Monte Carlo events. The other is

$${}^{\epsilon}\Psi_{b,b'}^{x} = \frac{1}{M_x}\sum_{i=1}^{M_x} {}^{\epsilon}A_b(\tau_i)\ {}^{\epsilon}A_{b'}^*(\tau_i)$$

for the detector accepted Monte Carlo events [1]. That is

$$\int {}^{\epsilon}A_b(\tau)\ {}^{\epsilon}A^*_{b'}(\tau)\eta(\tau)d\tau \approx \frac{1}{M}\sum_{i=1}^{M_x} {}^{\epsilon}A_b(\tau_i)\ {}^{\epsilon}A^*_{b'}(\tau_i) \tag{14}$$

$$= \eta_x\ {}^{\epsilon}\Psi^x_{b,b'}. \tag{15}$$

where the Monte Carlo efficiency η_x is defined as

$$\eta_x = M_x/M.$$

The extended log likelihood function which is maximized now becomes:

$$\ln L = \sum_{i=1}^{n} \ln[\sum_{\epsilon,b,b'} {}^{\epsilon}V_b\ {}^{\epsilon}V^*_{b'}\ {}^{\epsilon}A_b(\tau_i)\ {}^{\epsilon}A^*_{b'}(\tau_i)] - \eta_x \sum_{\epsilon,b,b'} {}^{\epsilon}V_b\ {}^{\epsilon}V^*_{b'}\ {}^{\epsilon}\Psi^x_{b,b'}.$$

In practice, the production are renormalized by setting

$$V \to \sqrt{\frac{n}{\eta_x}}V.$$

$\ln L$ can therefore be written

$$\ln L = \sum_{i=1}^{n} \ln[\sum_{\epsilon,b,b'} {}^{\epsilon}V_b\ {}^{\epsilon}V^*_{b'}\ {}^{\epsilon}A_b(\tau_i)\ {}^{\epsilon}A^*_{b'}(\tau_i)] - n \sum_{\epsilon,b,b'} {}^{\epsilon}V_b\ {}^{\epsilon}V^*_{b'}\ {}^{\epsilon}\Psi^x_{b,b'}.$$

It is advantageous to be renormalized since the renormalized production amplitudes are independent of variations in size among different bins, and therefore they are excellent starting values for neighboring bins [5]. The normalization condition, therefore, gives the form:

$$\sum_{\epsilon,b,b'} {}^{\epsilon}V_b\ {}^{\epsilon}V^{*\epsilon}_{b'}\Psi^x_{b,b'} = 1. \tag{16}$$

This term is useful to check the fitting codes. The predicted number of events is now given by

$$N = \frac{n}{\eta_x}\sum_{\epsilon,b,b'} {}^{\epsilon}V_b\ {}^{\epsilon}V^{*\epsilon}_{b'}\Psi_{b,b'}. \tag{17}$$

This equations shows that the acceptance corrected events N is proportional to the observed n divided by the Monte Carlo efficiency η_x.

For the data analysis we use the CERN program MINUIT to the minimization of $-\ln L$ which gives the maximization of $\ln L$.

References

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