Introduction to Dirac Equation II

Contents

1. Review of Dirac Equation

2. Free Motion of Dirac Particle

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1. Review of Dirac Equation

$$i\hbar\frac{\partial\psi}{\partial t}=(-i\hbar c\hat{\alpha}_k\nabla_k+\hat{\beta}mc^2)\psi\equiv\hat{H}\psi$$

1.1 K-G Eq. Compatiblity

⋆ **Correct energy-momentum relation for a relativistic free ptl.**

$E^2=p^2c^2 + m_0^2p^4$

→ satisfy K−G Equation : $-\hbar^2\frac{\partial^2\psi}{\partial^2 t} = (-\hbar^2c^2\nabla^2 + m_0^2c^4)\psi$

⋆ $\hat{\alpha},\hat{\beta}$ **should be hermetian. i.e** $\hat{\alpha}_i^+ = \hat{\alpha}_i,\ \hat{\beta}_i^+ = \hat{\beta}_i$

==> Find Out $\hat{\alpha}_k,\hat{\beta}$

$$\hat{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \hat{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\alpha}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \hat{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

1.2 Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

where $\rho = \psi \psi^{+} = \sum_{i=1}^{4} \psi_i^{*} \psi_i$, $\vec{j} = c \psi^{+} \hat{\alpha} \psi$

1.3Lorentz Covariance

2. Free Motion of Dirac Particle

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

$$= (c\hat{\alpha}\cdot\hat{p} + m_0c^2\hat{\beta})\psi$$

$$= (-i\hbar c\nabla\hat{\alpha} + m_0c^2\hat{\beta})\psi$$

⋆ **Stationary State :** $\psi(x,t) = \psi(x)e^{\frac{\epsilon t}{i\hbar}}$

Let's try 4 components spinor into two 2-component spinor ϕ, χ

==> $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}\phi\\ \chi\end{pmatrix}$ **where** $\phi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \chi = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$

==> $$\epsilon\begin{pmatrix}\phi\\ \chi\end{pmatrix} = c\begin{pmatrix}0 & \hat{\sigma}\\ \hat{\sigma} & 0\end{pmatrix}\cdot\vec{p}\begin{pmatrix}\phi\\ \chi\end{pmatrix} + m_0^2c^2\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix}$$

or

$$\epsilon\phi = c\vec{\sigma}\cdot\vec{p}\chi + m_0c^2\phi \ ,$$

$$\epsilon\chi = c\vec{\sigma}\cdot\vec{p}\phi - m_0c^2\chi$$

⋆with definite momentum p, $\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}\phi_0\\ \chi_0\end{pmatrix}e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$

==> $$(\epsilon - m_0c^2)I\phi_0 - c\vec{\sigma}\cdot\vec{p}\chi_0 = 0,$$

$$-c\vec{\sigma}\cdot\vec{p}\phi_0 + (\epsilon + m_0c^2)I\chi_0 = 0$$

: above equation has non-trivial solution only for

$$\begin{vmatrix}(\epsilon - m_0c^2)I & -c\vec{\sigma}\cdot p\\ -c\vec{\sigma}\cdot\vec{p} & (\epsilon + m_0c^2)I\end{vmatrix} = 0$$

==> $(\epsilon^2 - m_0^2 c^4)I - c^2(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) = 0$

==> $\epsilon^2 = m_0^2 c^4 + c^2 p^2$

==> $\epsilon = \pm E_p, \quad E_p = +c\sqrt{p^2 + m_0^2 c^2}$

==> $\chi_0 = \frac{c(\hat{\sigma}\cdot\hat{p})}{m_0 c^2 + \epsilon}\phi_0$ for fixed ϵ

==> $\Psi_{p\lambda} = N(\frac{c(\vec{\sigma}\cdot\vec{p})}{m_0 c^2 + \lambda E_p}U)\frac{\exp[i(\vec{p}\cdot\vec{x} - \lambda E_p t)/\hbar]}{\sqrt{2\pi\hbar^3}}$

where normalization factor N determined by

$\int \Psi^{+p\lambda}(x,t)\Psi_{p'\lambda'}(x,t)d^3x = \delta_{\lambda\lambda'}\delta(p-p')$

==> $N = \sqrt{\frac{(m_0 c^2 + \lambda E_p)}{2\lambda E_p}}$

==> $\epsilon = \lambda E$ with λ : eigen value of ± 1