Introduction
to
Dirac Equation

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Contents

1. Introduction

Consider the Schroudinger Eq. for a free ptl.

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$: not Lorentz Invaraint

☛ Attempts to modify the Hamiltonian so that Equation may hold relativity

☛ Klein-Gordon Equation

Derivation of Klein-Gordon Eq. by means of the Correspondence Principal

$$E -> i\hbar\frac{\partial}{\partial t}, \overrightarrow{p} => -i\hbar\overline{\nabla}$$

Non-Relativistic

$$E = \frac{p^2}{2m}$$

Relativistic

$$p^{\mu}p_{\mu} = \frac{E^2}{c^2} - |\overrightarrow{p}^2| = m^2c^2$$

☛ $E^2 = p^2c^2 + m^2p^4$

$$E = \sqrt{p^2c^2 + m^2p^4}$$

$$i\hbar\frac{\partial\psi}{\partial t} = \sqrt{-\hbar^2c^2\nabla^2 + m^2c^4} \quad ??$$

2. Klein-Godon Equation

2.1 Derivation

Expand above equation using Taylor Expansion.
However, it will not give symmetric derivatives of space and time.

☛ using $E = \sqrt{p^2c^2 + m^2p^4}$,

try $-\hbar^2 \frac{\partial^2}{\partial^2 t}\psi = (-\hbar^2 c^2 \nabla^2 + m^2 c^4)\psi$ instead.

☛ $[\nabla_\mu \nabla^\mu + (\frac{mc}{\hbar})^2]\psi = 0$: Klein-Godon Equation.

< Notation Invention >

$$x^\mu \equiv (x^0 = ct, \vec{r}\,)$$

$$\nabla_\mu \equiv \frac{\partial}{\partial x^\mu}$$

☛ $$\nabla_\mu \nabla^\mu = \frac{\partial^2}{\partial^2 t} - \nabla^2$$

covariant vector : $A_i{}' = \sum_j \frac{\partial x_i{}'}{\partial x_j} A_j$

contravarient vector on gradiant : $\frac{\partial \varphi'}{\partial x_i{}'} = \sum_j \frac{\partial \varphi}{\partial x_j} \frac{\partial x_j}{\partial x_i}$

2.2 Continuity Equation

$$\psi^*[\nabla_\mu \nabla^\mu + (\frac{mc}{\hbar})^2]\psi - \psi[\nabla_\mu \nabla^\mu + (\frac{mc}{\hbar})^2]\psi^* = 0$$

☛ $\nabla_\mu(\psi^* \nabla^\mu \psi - \psi \nabla^\mu \psi^*) \equiv \nabla_\mu j^\mu = 0$

: four-current density

Using Continuity Equation

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot \vec{j} = 0$$

set

$j_\mu = \frac{i\hbar}{2m_0}(\psi^* \nabla_\mu \psi - \psi \nabla_\mu \psi^*)$: four-current density

$\rho = \frac{i\hbar}{2m_0 c^2}(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t})$: probability density

Consideration 1 : ρ is not Positive definite

→It cannot directly interpreted as a probability density

→$e\rho(\vec{x},t)$ possibilly be considered as the corresponding charge density

Consideration 2 : 2nd order diff'l equation in time t.

→initial value of ψ and $\frac{\partial \psi}{\partial t}$ can have any value

3. Dirac Equation

3.1 Derivation

Introduce 1st order diff'l eq. for positive prob. density

$$i\hbar\frac{\partial\psi}{\partial t} = (-i\hbar c\alpha^k\nabla_k + \beta mc^2)\psi \equiv H\psi$$

find α^k and β where $k = 1,2,3$

why 1st order in $\frac{\partial}{\partial t}$?

-> to avoid (-) prob. density

why 1st order in ∇_k ?

-> for relavistic covariance

α^k and β must be hermetian matrices so that H is hermetian, which is necessary for (+) conserved prob. density to exist.

=> α^k and β will be $N\times N$ matrices,

with $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ ... \\ \psi_N \end{pmatrix}$: N-component column vector

3.2 Condition for α^k and β

< Requirements >

1) The component of ψ should satisfy K-G Eq. so that plane wave fullfil $E^2 = p^2c^2 + m^2p^4$

2) conserved 4-current $(\rho, c\vec{j})$ exists, of which 0-th component is (+) density

3) The eq. must be Lorentz covariant.

< Find out α^k and β >

From

$$i\hbar\frac{\partial\psi}{\partial t} = (-i\hbar c\alpha^k\nabla_k + \beta mc^2)\psi,$$

$$i\hbar\frac{\partial^2\psi}{\partial^2 t}$$

$$= -\hbar^2c^2\sum_{i,j}\frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i)\nabla_i\nabla_j\psi \quad + \frac{\hbar mc^3}{i}\sum_{i=1}^{3}(\alpha^i\beta + \beta\alpha^i)\nabla_i\psi + \beta^2m^2c^4\psi$$

By comparing above Eq. with K-G, obtain

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}I$$

$$\alpha^i\beta + \beta\alpha^i = 0$$

$$(\alpha^i)^2 = \beta^2 = I$$

3.3 Continuity Equation

$\psi^{\dagger}\,[-i\hbar c\alpha^{k}\nabla_{k}+\beta mc^{2}]$

$=> i\hbar\psi^{\dagger}\overleftrightarrow{\frac{\partial\psi}{\partial t}}=\frac{\hbar c}{i}\psi^{\dagger}\alpha^{i}\nabla_{i}\psi+mc^{2}\psi^{\dagger}\beta\psi$

and

$[-i\hbar c\alpha^{k}\nabla_{k}+\beta mc^{2}]^{\dagger}\psi$

$=> -i\hbar\,(\overleftrightarrow{\frac{\partial\psi^{\dagger}}{\partial t}})\psi=-\frac{\hbar c}{i}\nabla_{i}\psi^{\dagger}\,(\alpha^{i})^{\dagger}\psi+mc^{2}\psi^{\dagger}\beta^{\dagger}\psi$

where adjoint row vector to ψ : $\psi^{\dagger}=\;(\psi_{1}^{*},\psi_{2}^{*},...,\psi_{N}^{*})$

☛

$$\frac{\partial}{\partial t}(\psi^{\dagger}\psi)=\;-c\,[\psi^{\dagger}\alpha^{i}\nabla_{i}\psi+\nabla_{i}\psi^{\dagger}\alpha^{i\,\dagger}\psi]+\frac{mc^{2}}{\hbar}(\psi^{\dagger}\beta^{\dagger}\psi-\psi^{\dagger}\beta\psi)$$

$\rightarrow\alpha^{i\,\dagger}=\alpha^{i}\,,\qquad\beta=\beta^{\dagger}$

Density :

$$\rho\equiv\psi^{\dagger}\psi$$
$$=\sum_{\alpha=1}^{N}\psi_{\alpha}^{*}\psi$$

Current Density :

$$j^{k}\equiv c\psi^{\dagger}\alpha^{k}\psi$$

With the 0-th component j^{μ},

$j^{0}\equiv c\rho$,

we may define 4 -current density

$$j^{\mu} \equiv (j^0, j^k) = (c\rho, \vec{j}\,)$$

=> Continuity Equation : $\nabla_{\mu} j^{\mu} = 0$

ρ is (+) definite, and within the framework of single particle theory, can be given "probability density" interpretation.

3.4 Analysis of Dirac Equation

α, β are anti-commute => $\alpha^k = -\beta\alpha^k\beta$

$(\alpha^k)^2 = \beta^2 = 1$ => eigenvalues are ± 1 only.

Using the cyclic invariance of the trace,

$Tr(\alpha^k) = -Tr(\beta\alpha^k\beta) = -Tr(\alpha^k)$

$\therefore Tr(\alpha^k) = Tr(\beta) = 0$

The number of (+) and (-) eigenvalues must be equal, hence N = even !!

* N = 2 is not sufficient
  for 2 X 2 ; $I, \sigma_x, \sigma_y, \sigma_z$ contain only 3(not 4) mutally anti-commute matrices
  => N = 4 is the smallest possible dimension to realize the algebraic structure.

Particular represenation is

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where σ^i :Pauli Matrix, I : 2X2 Unit Matrix

ψ : “4-spinor" or "spinor"

$\psi^\dagger$: hermitian adjoint spinor

< Dirac Equation in Covariant Form >

Rewrite the Dirac Eq. in 4-vector notation

$$i\hbar\beta\nabla_0\psi = -i\hbar\beta\alpha^i\nabla_i\psi + mc\psi$$

Now, define the Dirac Matrices

$$\gamma^0 \equiv \beta \ ; \ \gamma^i = \beta\alpha^i$$

=> γ^0 is hermitian, $(\gamma^0)^2 = I$

γ^j : anti-hermitain $(\gamma^k)^\dagger = -\gamma^k$, $(\gamma^k)^2 = -1$

< Algebraic structure of the Dirac Matrices >

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I$$

The Dirac Eq. now, takes the form

$$i\hbar\beta\nabla_0\psi = -i\hbar\beta\alpha^i\nabla_i\psi + mc\psi$$

=> $(-i\gamma^\mu\nabla_\mu + \frac{mc}{\hbar})\psi = 0$