Calculation of Kaon Semi-leptonic Decay Form Factor using Staggered Fermions

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Cabibbo-Kobayashi-Maskawa Matrix Element, V_{us}

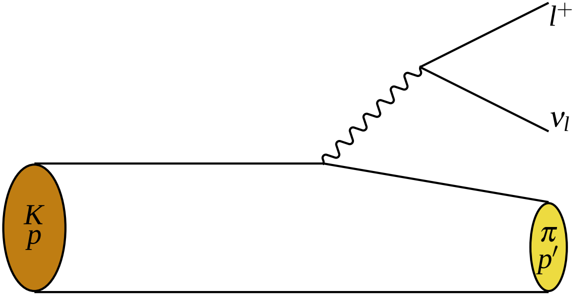
$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{1}$$

Unitarity on the first row

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \tag{2}$$

- To test the unitarity, it is most important to determine V_{us} accurately since V_{ub} is negligibly small, and V_{ud} has been determined in high precision.
- Precise determination of V_{us} is also important in the Wolfenstein parameterization. (CP violation)
- $|V_{us}|$ is traditionally obtained from the experimental rate for K_{l3} decays.

K_{l3} Decay



The decay rate is expressed by

$$\Gamma(K_{l3}) = \frac{G_F^2}{192\pi^3} M_K^5 C^2 I |V_{us}|^2 |f_+(0)|^2 S_{\rm ew} (1 + 2\Delta_{\rm SU(2)} + 2\Delta_{\rm em}). \quad (3)$$

$f_+(0)$ Vector form factor at zero momentum transfer,
$q^2 = (p' - p)^2 = 0$

Form Factors

$$\langle \pi(p')|V_\mu|K(p)\rangle = C\{(p_\mu + p'_\mu)f_+(q^2) + (p_\mu - p'_\mu)f_-(q^2)\} \tag{4}$$

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2) \tag{5}$$

V_μ Vector current operator, $\bar{s}\gamma_\mu u$.

C Clebsche-Gordan coefficient, (1 for neutral kaon; $1/\sqrt{2}$ for charged kaon)

$f_0(q^2)$ Scalar form factor. By definition, $f_0(0) = f_+(0)$.

- In the SU(3) symmetry limit ($m_u = m_d = m_s$), $|f_+(0)| = 1$ and $f_-(0) = 0$.
- The deviation of $f_+(0)$ from unity is of order $(m_s - \hat{m})^2$ by the Ademollo-Gatto theorem. ($\hat{m} = (m_u + m_d)/2$)

Lattice QCD

Euclidean Path Integral

$$\langle \mathscr{O} \rangle = \frac{1}{Z} \int \mathscr{D}[U]\mathscr{D}[\bar{\psi}]\mathscr{D}[\psi]\mathscr{O}[U,\psi,\bar{\psi}]e^{-S_G[U]-S_F[U,\psi,\bar{\psi}]}$$
$$= \frac{1}{Z} \int \mathscr{D}[U]e^{-S_{\text{eff}}[U]}A[U] \sum_{z_1\cdots z_n} \varepsilon^{z_1 z_2\cdots z_n}_{y_1 y_2\cdots y_n} M[U]^{-1}_{z_1 x_1} M[U]^{-1}_{z_2 x_2} \cdots M[U]^{-1}_{z_n x_n}$$

$$S_F = \bar{\psi}M[U]\psi \tag{6}$$

$$\mathscr{O}[U,\psi,\bar{\psi}] = \psi_{y_1}\bar{\psi}_{x_1}\psi_{y_2}\bar{\psi}_{x_2}\cdots\psi_{y_n}\bar{\psi}_{x_n}A[U] \tag{7}$$

$$S_{\text{eff}} = S_G[U] - \ln\det M[U] = S_G[U] - \operatorname{Tr}\ln M[U] \tag{8}$$

$$\langle \mathscr{O} \rangle = \lim_{T\to\infty} \frac{1}{Z_T} \operatorname{Tr}\left[e^{-\widehat{H}T}\mathscr{O}\right], \tag{9}$$

where $Z_T = \operatorname{Tr}[e^{-\widehat{H}T}]$.

Staggered Fermions

Staggered Fermion Action

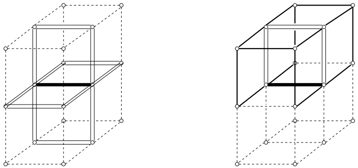
$$
\begin{aligned}
S_F =& a^4 \sum_n \Big\{ \sum_\mu \frac{1}{2a} \eta_\mu(n) \left[\bar{\chi}(n) U_\mu(n) \chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu}) U^\dagger_\mu(n) \chi(n) \right] \\
& + m \bar{\chi}(n) \chi(n) \Big\},
\end{aligned} \tag{10}
$$

where $\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu}$.

- Requires relatively low computational cost. Thus, it can be numerically simulated at lighter quark masses.
- But, it has 4 species per flavor. These are called “taste”.
- Furthermore, there are taste symmetry breaking at finite lattice spacing.
- For unimproved staggered fermions, this taste symmetry breaking results in significant discretization effects.

HYP smearing

HYP improved staggered fermion action are made by replacing the original gauge links by the gauge links fattened through 3-time smearing and 3 SU(3) projections as below.



$$V_{i,\mu} = \mathrm{Proj}_{SU(3)}\left[(1-\alpha_1)U_{i,\mu} + \frac{\alpha_1}{6}\sum_{\pm\nu\neq\mu}\tilde{V}_{i,\nu;\mu}\tilde{V}_{i+\hat{\nu},\mu;\nu}\tilde{V}^{\dagger}_{i+\hat{\mu},\nu;\mu}\right], \quad (11a)$$

$$\tilde{V}_{i,\mu;\nu} = \mathrm{Proj}_{SU(3)}\left[(1-\alpha_2)U_{i,\mu} + \frac{\alpha_2}{4}\sum_{\pm\rho\neq\nu,\mu}\bar{V}_{i,\rho;\nu\mu}\bar{V}_{i+\hat{\rho},\mu;\rho\nu}\bar{V}^{\dagger}_{i+\hat{\mu},\rho;\nu\mu}\right], \quad (11b)$$

$$\bar{V}_{i,\mu;\nu\rho} = \mathrm{Proj}_{SU(3)}\left[(1-\alpha_3)U_{i,\mu} + \frac{\alpha_3}{2}\sum_{\pm\eta\neq\rho,\nu,\mu}U_{i,\eta}U_{i+\hat{\eta},\mu}U^{\dagger}_{i+\hat{\mu},\eta}\right]. \quad (11c)$$

Comparison between Asqtad and HYP Valence Quarks

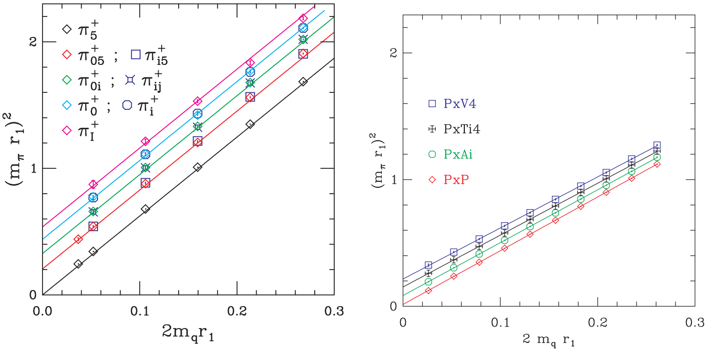


Figure: $(r_1 m_\pi)^2$ vs. $2r_1 m_q$ on unquenched configurations. Left: with asqtad valence quarks; Right: with HYP-smeared valence staggered fermions (shows only LT tastes).

HYP staggered fermion fermions reduce the splittings by a factor of 2.5–3 in comparison with the asqtad quarks.

Correlation Functions

The hadronic matrix elements are obtained from calculating the relevant correlation functions.

$$\begin{aligned} C_\mu^{PQ}(t_1,t_2;\boldsymbol{k}_1,\boldsymbol{k}_3) = & \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3} \langle \mathscr{O}_Q(\boldsymbol{x}_1,t_1+t_0) V_\mu(\boldsymbol{x}_2,t_2+t_0) \mathscr{O}_P^\dagger(\boldsymbol{x}_3,t_0)\rangle \\ & \times e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1} e^{i(\boldsymbol{k}_3-\boldsymbol{k}_1)\cdot\boldsymbol{x}_2} e^{-i\boldsymbol{k}_3\cdot\boldsymbol{x}_3} \\ & \xrightarrow[t_2,(t_1-t_2)\to\infty]{} V_s \frac{Z_P^* Z_Q}{4E_Q(\boldsymbol{k}_1)E_P(\boldsymbol{k}_3)} \langle Q(\boldsymbol{k}_1)|V_\mu|P(\boldsymbol{k}_3)\rangle \\ & \times e^{-E_Q(\boldsymbol{k}_1)(t_1-t_2)} e^{-E_P(\boldsymbol{k}_3)t_2}. \end{aligned} \tag{12}$$

$$\begin{aligned} C^P(t;\boldsymbol{k}_1) = & \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2} \langle \mathscr{O}_P(\boldsymbol{x}_1,t+t_0) \mathscr{O}_P^\dagger(\boldsymbol{x}_2,t_0)\rangle e^{i\boldsymbol{k}_1\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)} \\ & \xrightarrow[t\to\infty]{} V_s \frac{Z_P^* Z_P}{2E_P(\boldsymbol{k}_1)} e^{-E_P(\boldsymbol{k}_1)t} \end{aligned} \tag{13}$$

P and Q are either kaon or pion.

Interpolating Operators

In these calculations of the correlation functions, we use the following interpolating operators with staggered fermion fields.

Pseudo-scalar Operator

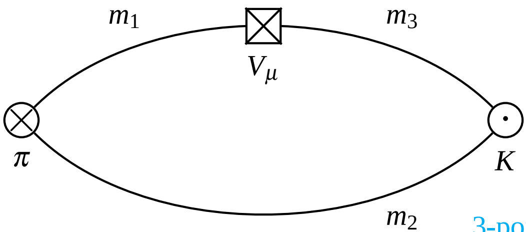
$$P(t,\boldsymbol{k}) = \sum_{\boldsymbol{n}} \varepsilon(n)\bar{\chi}(n)\chi(n)e^{i\boldsymbol{k}\cdot\boldsymbol{n}}, \quad \varepsilon(n) = (-1)^{\Sigma_\nu n_\nu} \tag{14}$$

Vector Current Operator

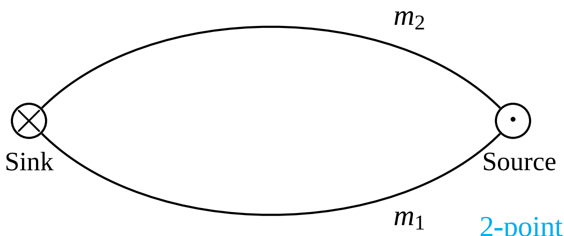
$$V_\mu(t,\boldsymbol{k}) = \frac{1}{2}\sum_{\boldsymbol{n}} e^{i\boldsymbol{k}\cdot\boldsymbol{n}}\eta_\mu(n)\left[\bar{\chi}(n)U_\mu(n)\chi(n+\hat{\mu}) + \bar{\chi}(n+\hat{\mu})U_\mu^\dagger(n)\chi(n)\right] \tag{15}$$

Diagrams of contractions of quark propagators

A correlation function is a contraction of quark propagators generating from sources.



3-point function



2-point function

Noise U(1) momentum sources and PxP insertion sources

To generate quark with non-zero momentum, we use Noise U(1) momentum sources. In the case of the 3-point functions, we need to insert $(\gamma_5 \otimes \gamma_5)$ operator into quark propagators and make new sources. These sources are called PxP insertion sources.

Noise U(1) momentum sources

$$h([\boldsymbol{n},t],b;t';p) = \delta_{t,t'}\eta(\boldsymbol{n},b)e^{i\boldsymbol{p}\cdot\boldsymbol{n}} \tag{16}$$

PxP insertion sources

$$h_2(n,c;t_3;\boldsymbol{k}) = \delta_{t,t_3}\psi_2(n,c;t_1;\boldsymbol{0})\varepsilon(n)e^{i\boldsymbol{k}\cdot\boldsymbol{n}}; \quad n = (\boldsymbol{n},t) \tag{17}$$

The quark propagators are obtained by solving the Dirac equation with these sources.

$$\sum_{y,b}[\not{D} + m_f](x,a;y,b)\psi_f(y,b;t';\boldsymbol{k}) = h(x,a;t';\boldsymbol{k}) \tag{18}$$

Double Ratio Method

To determine $f_+(0)$ precisely, we adopt a strategy employed in C. Dawson et al. Phys. Rev., D74:114502, 2006. This is called "double ratio method".

$$
\begin{aligned}
R(t) &= \frac{C_4^{K\pi}(t,t';\mathbf{0},\mathbf{0})C_4^{\pi K}(t,t';\mathbf{0},\mathbf{0})}{C_4^{KK}(t,t';\mathbf{0},\mathbf{0})C_4^{\pi\pi}(t,t';\mathbf{0},\mathbf{0})} \\
&\xrightarrow[t,(t'-t)\to\infty]{} \frac{\langle\pi(\mathbf{0})|V_4|K(\mathbf{0})\rangle\langle K(\mathbf{0})|V_4|\pi(\mathbf{0})\rangle}{\langle K(\mathbf{0})|V_4|K(\mathbf{0})\rangle\langle\pi(\mathbf{0})|V_4|\pi(\mathbf{0})\rangle} \\
&= \frac{(M_K+M_\pi)^2}{4M_K M_\pi}\left|f_0(q^2_{\mathrm{max}})\right|^2
\end{aligned}
\tag{19}
$$

- $q^2_{max} = (M_K - M_\pi)^2$

$q^2 = 0$ Interpolation

On the lattice, momenta are quantized so that we cannot obtain $f_0(q^2)$ at point $q^2 = 0$. Thus, we must interpolate the $f_0(q^2)$ data to $q^2 = 0$. There are three kinds of fit.

- Linear fit : $f_0(q^2) = f_0(0)(1 + \lambda_0^{(1)} q^2)$
- quadratic fit : $f_0(q^2) = f_0(0)(1 + \lambda_0^{(1)} q^2 + \lambda_0^{(2)} q^4)$
- pole fit : $f_0(q^2) = f_0(0)/(1 - \lambda_0^{(1)} q^2)$

Another Method

From the equation of motion,

$$\partial_\mu V^\mu(x) = (m_s - m_u)S(x), \tag{20}$$

where $V^\mu(x) = \bar{s}(x)\gamma^\mu u(x)$ and $m_s(m_u)$ is the mass of the strange(up) quark mass. $S(x)$ is the scalar operator $\bar{s}(x)u(x)$. A hadronic matrix element of $V^\mu(x)$

$$\langle \pi(p')|V^\mu(x)|K(p)\rangle = e^{ix(p'-p)}\langle \pi(p')|V^\mu(0)|K(p)\rangle, \tag{21}$$

from translation invariance.

$$iq_\mu\langle \pi(p')|V^\mu(x)|K(p)\rangle = (m_s - m_u)\langle \pi(p')|S(x)|K(p)\rangle, \tag{22}$$

where $q_\mu = p'_\mu - p_\mu$.

Form Factors

$$\langle \pi(p')|V^{\mu}(0)|K(p)\rangle = f_{+}(q^2)\left[p^{\mu} + p'^{\mu} - \frac{m_K^2 - m_\pi^2}{q^2} q^{\mu}\right] + f_0(q^2)\frac{m_K^2 - m_\pi^2}{q^2} q^{\mu}, \tag{23}$$

$$\langle \pi(p')|S(0)|K(p)\rangle = \frac{m_K^2 - m_\pi^2}{m_s - m_u} f_0(q^2). \tag{24}$$

Thus, we can calculate the scalar form factor $f_0(q^2)$ from the matrix element of the scalar operator;

$$f_0(q^2) = \frac{m_s - m_u}{m_K^2 - m_\pi^2} \langle \pi(p')|S|K(p)\rangle. \tag{25}$$

Because $f_{+}(0) = f_0(0)$,

$$f_{+}(0) = \frac{m_s - m_u}{m_K^2 - m_\pi^2} \langle \pi(p')|S|K(p)\rangle_{q^2=0}. \tag{26}$$

Scalar Operator

Scalar Operator

$$S(t,\boldsymbol{k}) = \sum_{\boldsymbol{n},c} \bar{\chi}^c(n)\chi^c(n)e^{i\boldsymbol{k}\cdot\boldsymbol{n}} \tag{27}$$

Scalar Operator Three-point Functions

$$\begin{aligned} C_I^{K\pi}(t_1,t_2;\boldsymbol{k}_1,\boldsymbol{k}_3) &= \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3} \langle \mathscr{O}_\pi(\boldsymbol{x}_1,t_1+t_0)S(\boldsymbol{x}_2,t_2+t_0)\mathscr{O}_K^\dagger(\boldsymbol{x}_3,t_0)\rangle \\ &\quad \times e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1}e^{i(\boldsymbol{k}_3-\boldsymbol{k}_1)\cdot\boldsymbol{x}_1}e^{i\boldsymbol{k}_3\cdot\boldsymbol{x}} \\ &\xrightarrow[t,(t'-t)\to\infty]{} V_s\frac{Z_K^* Z_\pi}{4E_K(\boldsymbol{k}_3)E_\pi(\boldsymbol{k}_1)} \\ &\quad \times \langle \boldsymbol{\pi}(\boldsymbol{k}_1)|S|K(\boldsymbol{k}_3)\rangle e^{-E_K(\boldsymbol{k}_3)t_2-E_\pi(\boldsymbol{k}_1)(t_1-t_2)} \end{aligned} \tag{28}$$

Partially twisted boundary condition

Under periodic boundary condition

$$\psi(x+\hat{e}_i L) = \psi(x), \quad i = 1, 2, 3, \tag{29}$$

momenta are quantized as $\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{n}$.
If we twist the boundary condition by θ:

$$\psi(x+\hat{e}_i L) = e^{i\theta_i}\psi(x), \quad i = 1, 2, 3, \tag{30}$$

we can obtain the momenta as

$$p_i = \frac{\theta_i}{L} + \frac{2\pi}{L} n_i, \quad i = 1, 2, 3. \tag{31}$$