스태거드 페르미온을 사용한 K_{l3} 붕괴 형태인자의 계산

측정 코드 테스트 방법 및 결과

배태길

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Cabibbo-Kobayashi-Maskawa Matrix Element, V_{us}

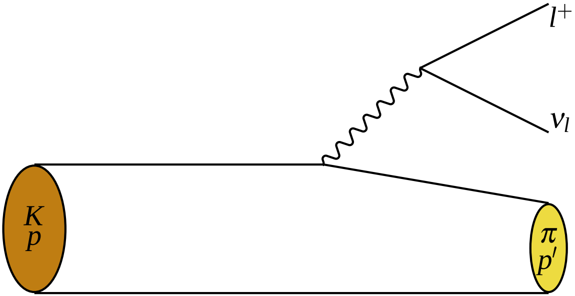
$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{1}$$

Unitarity on the first row

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \tag{2}$$

- To test the unitarity, it is most important to determine V_{us} accurately since V_{ub} is negligibly small, and V_{ud} has been determined in high precision.
- Precise determination of V_{us} is also important in the Wolfenstein parameterization. (CP violation)
- $|V_{us}|$ is traditionally obtained from the experimental rate for K_{l3} decays.

K_{l3} Decay



The decay rate is expressed by

$$\Gamma(K_{l3}) = \frac{G_F^2}{192\pi^3} M_K^5 C^2 I |V_{us}|^2 |f_+(0)|^2 S_{\text{ew}} (1 + 2\Delta_{\text{SU}(2)} + 2\Delta_{\text{em}}). \quad (3)$$

$f_+(0)$ Vector form factor at zero momentum transfer, $q^2 = (p' - p)^2 = 0$

Form Factors

$$\langle \pi(p')|V_\mu|K(p)\rangle = C\{(p_\mu + p'_\mu)f_+(q^2) + (p_\mu - p'_\mu)f_-(q^2)\} \qquad (4)$$

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2) \qquad (5)$$

V_μ Vector current operator, $\bar{s}\gamma_\mu u$.

C Clebsche-Gordan coefficient, (1 for neutral kaon; $1/\sqrt{2}$ for charged kaon)

$f_0(q^2)$ Scalar form factor. By definition, $f_0(0) = f_+(0)$.

- In the SU(3) symmetry limit ($m_u = m_d = m_s$), $|f_+(0)| = 1$ and $f_-(0) = 0$.
- The deviation of $f_+(0)$ from unity is of order $(m_s - \hat{m})^2$ by the Ademollo-Gatto theorem. ($\hat{m} = (m_u + m_d)/2$)

Lattice QCD

Euclidean Path Integral

$$\begin{aligned}\langle \mathscr{O} \rangle &= \frac{1}{Z} \int \mathscr{D}[U]\mathscr{D}[\bar{\psi}]\mathscr{D}[\psi]\mathscr{O}[U,\psi,\bar{\psi}]e^{-S_G[U]-S_F[U,\psi,\bar{\psi}]} \\ &= \frac{1}{Z} \int \mathscr{D}[U]e^{-S_{\text{eff}}[U]}A[U] \sum_{z_1\cdots z_n} \varepsilon^{z_1 z_2 \cdots z_n}_{y_1 y_2 \cdots y_n} M[U]^{-1}_{z_1 x_1} M[U]^{-1}_{z_2 x_2} \cdots M[U]^{-1}_{z_n x_n}\end{aligned}$$

$$S_F = \bar{\psi} M[U] \psi \tag{6}$$

$$\mathscr{O}[U,\psi,\bar{\psi}] = \psi_{y_1} \bar{\psi}_{x_1} \psi_{y_2} \bar{\psi}_{x_2} \cdots \psi_{y_n} \bar{\psi}_{x_n} A[U] \tag{7}$$

$$S_{\text{eff}} = S_G[U] - \ln\det M[U] = S_G[U] - \operatorname{Tr}\ln M[U] \tag{8}$$

$$\langle \mathscr{O} \rangle = \lim_{T\to\infty} \frac{1}{Z_T} \operatorname{Tr}\left[e^{-\widehat{H}T}\mathscr{O}\right], \tag{9}$$

where $Z_T = \operatorname{Tr}[e^{-\widehat{H}T}]$.

Staggered Fermions

Staggered Fermion Action

$$S_F = a^4 \sum_n \Big\{ \sum_\mu \frac{1}{2a} \eta_\mu(n) \big[\bar{\chi}(n) U_\mu(n) \chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu}) U_\mu^\dagger(n) \chi(n) \big] + m \bar{\chi}(n) \chi(n) \Big\}, \tag{10}$$

where $\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu}$.

- Requires relatively low computational cost. Thus, it can be numerically simulated at lighter quark masses.
- But, it has 4 species per flavor. These are called “taste”.
- Furthermore, there are taste symmetry breaking at finite lattice spacing.
- For unimproved staggered fermions, this taste symmetry breaking results in significant discretization effects.

Correlation Functions

The hadronic matrix elements are obtained from calculating the relevant correlation functions.

$$\begin{aligned} C_{\mu}^{PQ}(t_1, t_2; \boldsymbol{k}_1, \boldsymbol{k}_3) &= \sum_{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3} \langle \mathscr{O}_Q(\boldsymbol{x}_1, t_1 + t_0) V_{\mu}(\boldsymbol{x}_2, t_2 + t_0) \mathscr{O}_P^{\dagger}(\boldsymbol{x}_3, t_0) \rangle \\ &\quad \times e^{i\boldsymbol{k}_1 \cdot \boldsymbol{x}_1} e^{i(\boldsymbol{k}_3 - \boldsymbol{k}_1) \cdot \boldsymbol{x}_2} e^{-i\boldsymbol{k}_3 \cdot \boldsymbol{x}_3} \\ &\xrightarrow[t_2, (t_1 - t_2) \to \infty]{} V_s \frac{Z_P^* Z_Q}{4E_Q(\boldsymbol{k}_1) E_P(\boldsymbol{k}_3)} \langle Q(\boldsymbol{k}_1) | V_{\mu} | P(\boldsymbol{k}_3) \rangle \\ &\quad \times e^{-E_Q(\boldsymbol{k}_1)(t_1 - t_2)} e^{-E_P(\boldsymbol{k}_3) t_2}. \end{aligned} \tag{11}$$

$$\begin{aligned} C^P(t; \boldsymbol{k}_1) &= \sum_{\boldsymbol{x}_1, \boldsymbol{x}_2} \langle \mathscr{O}_P(\boldsymbol{x}_1, t + t_0) \mathscr{O}_P^{\dagger}(\boldsymbol{x}_2, t_0) \rangle e^{i\boldsymbol{k}_1 \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)} \\ &\xrightarrow[t \to \infty]{} V_s \frac{Z_P^* Z_P}{2E_P(\boldsymbol{k}_1)} e^{-E_P(\boldsymbol{k}_1) t} \end{aligned} \tag{12}$$

P and Q are either kaon or pion.

Interpolating Operators

In these calculations of the correlation functions, we use the following interpolating operators with staggered fermion fields.

Pseudo-scalar Operator

$$P(t,\boldsymbol{k}) = \sum_{\boldsymbol{n}} \varepsilon(n)\bar{\chi}(n)\chi(n)e^{i\boldsymbol{k}\cdot\boldsymbol{n}}, \quad \varepsilon(n) = (-1)^{\sum_\nu n_\nu} \tag{13}$$

Vector Current Operator

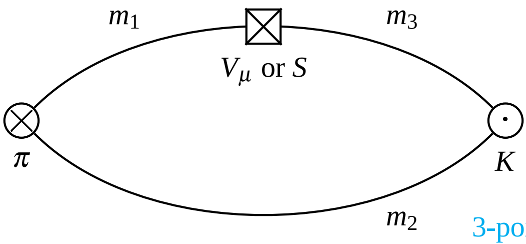
$$V_\mu(t,\boldsymbol{k}) = \frac{1}{2}\sum_{\boldsymbol{n}} e^{i\boldsymbol{k}\cdot\boldsymbol{n}}\eta_\mu(n)\left[\bar{\chi}(n)U_\mu(n)\chi(n+\hat{\mu}) + \bar{\chi}(n+\hat{\mu})U_\mu^\dagger(n)\chi(n)\right] \tag{14}$$

Scalar Operator

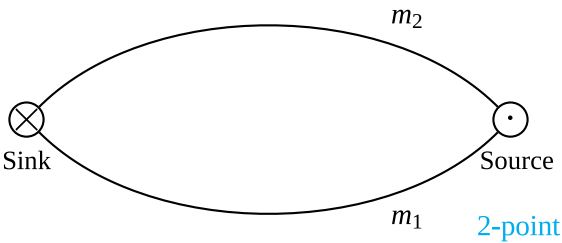
$$S(t,\boldsymbol{k}) = \sum_{\boldsymbol{n},c} \bar{\chi}^c(n)\chi^c(n)e^{i\boldsymbol{k}\cdot\boldsymbol{n}} \tag{15}$$

Diagrams of contractions of quark propagators

A correlation function is a contraction of quark propagators generated from sources.



3-point function



2-point function

Noise U(1) momentum sources and PxP insertion sources

To generate quark with non-zero momentum, we use Noise U(1) momentum sources. In the case of the 3-point functions, we need to insert $(\gamma_5 \otimes \gamma_5)$ operator into quark propagators and make new sources. These sources are called PxP insertion sources.

Noise U(1) momentum sources

$$h([\boldsymbol{n},t],b;t';p) = \delta_{t,t'}\eta(\boldsymbol{n},b)e^{i\boldsymbol{p}\cdot\boldsymbol{n}} \qquad (16)$$

PxP insertion sources

$$h_2(n,c;t_3;\boldsymbol{k}) = \delta_{t,t_3}\psi_2(n,c;t_1;\boldsymbol{0})\varepsilon(n)e^{i\boldsymbol{k}\cdot\boldsymbol{n}}; \quad n = (\boldsymbol{n},t) \qquad (17)$$

The quark propagators are obtained by solving the Dirac equation with these sources.

$$\sum_{y,b}[\not{D} + m_f](x,a;y,b)\psi_f(y,b;t';\boldsymbol{k}) = h(x,a;t';\boldsymbol{k}) \qquad (18)$$

Gauge Invariance Test

Under SU(3) gauge transformation $g(x) \in SU(3)$,

$$U_\mu(x) \rightarrow g(x) U_\mu(x) g^\dagger(x+\hat{\mu}) \tag{19}$$

$$\psi(x) \rightarrow g(x) \psi(x) \tag{20}$$

Random SU(3) gauge transformation over all lattice sites except for the U(1) noise source time slice.

Random SU(3) matrix

Ref. : Maris Ozols, *How to generate a random unitary matrix.*

Random SU(2) matrices

$$SU(2) = \begin{pmatrix} e^{i\psi}\cos\phi & e^{i\chi}\sin\phi \\ -e^{-i\chi}\sin\phi & e^{-i\psi}\cos\phi \end{pmatrix} \tag{21}$$

Generate three random numbers: $\psi, \chi \in [0, 2\pi]$ and $\xi \in [0, 1]$

$\phi = \arcsin\sqrt{\xi} \rightarrow$ Uniformly distributed SU(2) matrices

Givens rotation $G(i,j)$: Rotate sub-matrix ((i,i), (i, j); (j, i), (j, j)) of 3x3 identity matrix by the generated random SU(2) matrix

$$G(i,j,\theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & -s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

Random SU(3) matrix $= G_1(1,2)G_2(2,3)G_3(3,1)$

Ward Identity test

$$\partial_\mu V_\mu = (m_s - m_u)S \tag{22}$$

An Ward identity :

$$iq_\mu \langle \pi(p')|V_\mu|K(p)\rangle = (m_s - m_u)\langle \pi(p')|S|K(p)\rangle \tag{23}$$

Our correlation functions $C_\mu^{K\pi}(t,\boldsymbol{k})$ and $C_I^{K\pi}(t,\boldsymbol{k})$ correspond to the summations of $\langle \pi(k)|V_\mu|K(\boldsymbol{p}=\boldsymbol{0})\rangle$ and $\langle \pi(k)|S|K(\boldsymbol{p}=\boldsymbol{0})\rangle$ over whole space volume, respectively.
For $\boldsymbol{k}=0$,

$$\partial_4 \langle \pi(0)|V^4|K(\boldsymbol{p}=\boldsymbol{0})\rangle = (m_s - m_u)\langle \pi(0)|S|K(0)\rangle. \tag{24}$$

We check that for $\boldsymbol{k}=0$, $C_4^{K\pi}(t) - C_4^{K\pi}(t-1) = (m_s - m_u)C_I^{K\pi}(t)$.

Ward Identity test

```
$ ward_id.pl k2pi_ff.vector3p.t26t60.m0.00500.m0.01000.m0.05000.kx0ky0kz0.
 k2pi_mu4.dat.30 k2pi_ff.scalar3p.t26t60.m0.00500.m0.01000.m0.05000.kx0ky0kz0
 .k2pi_I.dat.30
t1 = 26, t2 = 60
======== REAL PART ========
t       C_4(t)-C_4(t-1) C_I(t)*(m_s-m_u) diff
01 -1.3899336 -1.3899336 +4.024711e-11
02 -1.1607124 -1.1607124 +2.260278e-10
....
24 -0.034726026 -0.034726026 +2.061525e-11
25 -0.028316772 -0.028316772 +3.851502e-10
26 -0.17623825 -0.026272962 +8.509236e-01
27 -0.023659762 -0.023659762 +1.104351e-10
28 -0.02119814 -0.02119814 +5.400341e-11
.....
58 -2.2002058 -2.2002058 -1.610474e-10
59 -1.9810066 -1.9810066 +2.122749e-10
60 +28.494561 -2.5367882 +1.089027e+00
61  -2.155153  -2.155153 +1.327788e-10
...
```

The disagreements in t = 26 and t = 60 are caused by the contact term.