# Calculation of Kaon Semi-leptonic Decay Form Factor using HYP-smeared Staggered Fermions

배태길

슈퍼컴퓨팅본부,
고에너지물리연구팀

사월 열셋째날 – 열다섯째날, 이천열한번째해
한국물리학회 봄 학술논문발표회,
대전컨벤션센터

# Cabibbo-Kobayashi-Maskawa Matrix Element, $V_{us}$

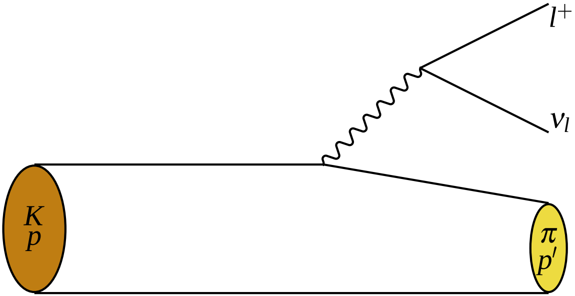
$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{1}$$

## Unitarity on the first row

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \tag{2}$$

- To test the unitarity, it is most important to determine $V_{us}$ accurately since $V_{ub}$ is negligibly small, and $V_{ud}$ has been determined in high precision.
- Precise determination of $V_{us}$ is also important in the Wolfenstein parameterization. (CP violation)
- $|V_{us}|$ is traditionally obtained from the experimental rate for $K_{l3}$ decays.

# $K_{l3}$ Decay




The decay rate is expressed by

$$\Gamma(K_{l3}) = \frac{G_F^2}{192\pi^3} M_K^5 C^2 I |V_{us}|^2 |f_+(0)|^2 S_{\text{ew}} (1 + 2\Delta_{\text{SU}(2)} + 2\Delta_{\text{em}}). \quad (3)$$

$f_+(0)$ Vector form factor at zero momentum transfer, $q^2 = (p' - p)^2 = 0$

# Form Factors

$$\langle \pi(p')|V_\mu|K(p)\rangle = C\{(p_\mu + p'_\mu)f_+(q^2) + (p_\mu - p'_\mu)f_-(q^2)\} \quad (4)$$

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2) \quad (5)$$

$V_\mu$ Vector current operator, $\bar{s}\gamma_\mu u$.

$C$ Clebsche-Gordan coefficient, (1 for neutral kaon; $1/\sqrt{2}$ for charged kaon)

$f_0(q^2)$ Scalar form factor. By definition, $f_0(0) = f_+(0)$.

- In the SU(3) symmetry limit ($m_u = m_d = m_s$), $|f_+(0)| = 1$ and $f_-(0) = 0$.
- The deviation of $f_+(0)$ from unity is of order $(m_s - \hat{m})^2$ by the Ademollo-Gatto theorem. ($\hat{m} = (m_u + m_d)/2$)

# Lattice QCD

## Euclidean Path Integral

$$\langle\mathscr{O}\rangle = \frac{1}{Z}\int \mathscr{D}[U]\mathscr{D}[\bar\psi]\mathscr{D}[\psi]\mathscr{O}[U,\psi,\bar\psi]e^{-S_G[U]-S_F[U,\psi,\bar\psi]}$$
$$= \frac{1}{Z}\int \mathscr{D}[U]e^{-S_{\rm eff}[U]}A[U]\sum_{z_1\cdots z_n}\varepsilon^{z_1z_2\cdots z_n}_{y_1y_2\cdots y_n}M[U]^{-1}_{z_1x_1}M[U]^{-1}_{z_2x_2}\cdots M[U]^{-1}_{z_nx_n}$$

$$S_F = \bar\psi M[U]\psi \tag{6}$$

$$\mathscr{O}[U,\psi,\bar\psi] = \psi_{y_1}\bar\psi_{x_1}\psi_{y_2}\bar\psi_{x_2}\cdots\psi_{y_n}\bar\psi_{x_n}A[U] \tag{7}$$

$$S_{\rm eff} = S_G[U] - \ln\det M[U] = S_G[U] - {\rm Tr}\ln M[U] \tag{8}$$

$$\langle\mathscr{O}\rangle = \lim_{T\to\infty}\frac{1}{Z_T}{\rm Tr}\left[e^{-\widehat{H}T}\mathscr{O}\right], \tag{9}$$

where $Z_T = {\rm Tr}[e^{-\widehat{H}T}]$.

# Staggered Fermions

## Staggered Fermion Action

$$
\begin{aligned}
S_F =& a^4 \sum_n \Big\{ \sum_\mu \frac{1}{2a} \eta_\mu(n) \left[ \bar{\chi}(n) U_\mu(n) \chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu}) U_\mu^\dagger(n) \chi(n) \right] \\
&+ m \bar{\chi}(n) \chi(n) \Big\},
\end{aligned} \tag{10}
$$

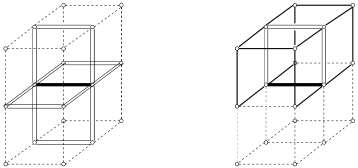
where $\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu}$.

- Requires relatively low computational cost. Thus, it can be numerically simulated at lighter quark masses.
- But, it has 4 species per flavor. These are called "taste".
- Furthermore, there are taste symmetry breaking at finite lattice spacing.
- For unimproved staggered fermions, this taste symmetry breaking results in significant discretization effects.

# HYP smearing

HYP improved staggered fermion action are made by replacing the original gauge links by the gauge links fattened through 3-time smearing and 3 SU(3) projections as below.

a) b)



$$V_{i,\mu} = \mathrm{Proj}_{SU(3)}\left[(1-\alpha_1)U_{i,\mu} + \frac{\alpha_1}{6}\sum_{\pm\nu\neq\mu}\tilde{V}_{i,\nu;\mu}\tilde{V}_{i+\hat{\nu},\mu;\nu}\tilde{V}^{\dagger}_{i+\hat{\mu},\nu;\mu}\right], \quad (11a)$$

$$\tilde{V}_{i,\mu;\nu} = \mathrm{Proj}_{SU(3)}\left[(1-\alpha_2)U_{i,\mu} + \frac{\alpha_2}{4}\sum_{\pm\rho\neq\nu,\mu}\bar{V}_{i,\rho;\nu\mu}\bar{V}_{i+\hat{\rho},\mu;\rho\nu}\bar{V}^{\dagger}_{i+\hat{\mu},\rho;\nu\mu}\right], \quad (11b)$$

$$\bar{V}_{i,\mu;\nu\rho} = \mathrm{Proj}_{SU(3)}\left[(1-\alpha_3)U_{i,\mu} + \frac{\alpha_3}{2}\sum_{\pm\eta\neq\rho,\nu,\mu}U_{i,\eta}U_{i+\hat{\eta},\mu}U^{\dagger}_{i+\hat{\mu},\eta}\right]. \quad (11c)$$

# Comparison between Asqtad and HYP Valence Quarks

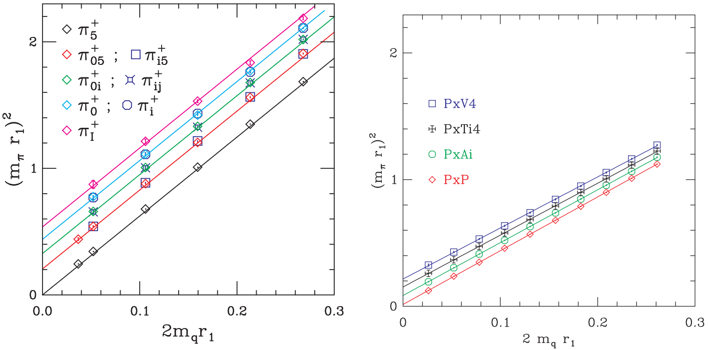



Figure: $(r_1 m_\pi)^2$ vs. $2r_1 m_q$ on unquenched configurations. Left: with asqtad valence quarks; Right: with HYP-smeared valence staggered fermions (shows only LT tastes).

HYP staggered fermion fermions reduce the splittings by a factor of 2.5–3 in comparison with the asqtad quarks.

# Correlation Functions

The hadronic matrix elements are obtained from calculating the relevant correlation functions.

$$\begin{aligned} C_\mu^{PQ}(t_1,t_2;\boldsymbol{k}_1,\boldsymbol{k}_3) &= \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3} \langle \mathcal{O}_Q(\boldsymbol{x}_1,t_1+t_0) V_\mu(\boldsymbol{x}_2,t_2+t_0) \mathcal{O}_P^\dagger(\boldsymbol{x}_3,t_0)\rangle \\ &\qquad \times e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1} e^{i(\boldsymbol{k}_3-\boldsymbol{k}_1)\cdot\boldsymbol{x}_2} e^{-i\boldsymbol{k}_3\cdot\boldsymbol{x}_3} \\ &\xrightarrow[t_2,(t_1-t_2)\to\infty]{} V_s \frac{Z_P^* Z_Q}{4E_Q(\boldsymbol{k}_1)E_P(\boldsymbol{k}_3)} \langle Q(\boldsymbol{k}_1)|V_\mu|P(\boldsymbol{k}_3)\rangle \\ &\qquad \times e^{-E_Q(\boldsymbol{k}_1)(t_1-t_2)} e^{-E_P(\boldsymbol{k}_3)t_2}. \end{aligned} \tag{12}$$

$$\begin{aligned} C^P(t;\boldsymbol{k}_1) &= \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2} \langle \mathcal{O}_P(\boldsymbol{x}_1,t+t_0) \mathcal{O}_P^\dagger(\boldsymbol{x}_2,t_0)\rangle e^{i\boldsymbol{k}_1\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)} \\ &\xrightarrow[t\to\infty]{} V_s \frac{Z_P^* Z_P}{2E_P(\boldsymbol{k}_1)} e^{-E_P(\boldsymbol{k}_1)t} \end{aligned} \tag{13}$$

$P$ and $Q$ are either kaon or pion.

# Interpolating Operators

In these calculations of the correlation functions, we use the following interpolating operators with staggered fermion fields.

## Pseudo-scalar Operator

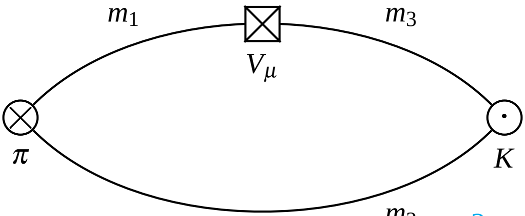
$$P(t,\boldsymbol{k}) = \sum_{\boldsymbol{n}} \varepsilon(n)\bar{\chi}(n)\chi(n)e^{i\boldsymbol{k}\cdot\boldsymbol{n}}, \quad \varepsilon(n) = (-1)^{\Sigma_\nu n_\nu} \tag{14}$$

## Vector Current Operator

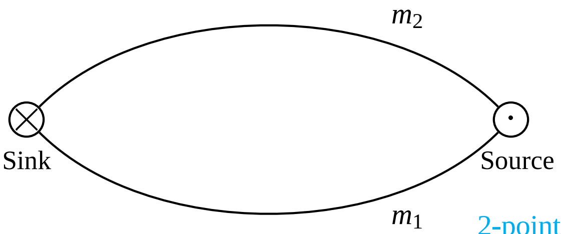
$$V_\mu(t,\boldsymbol{k}) = \frac{1}{2}\sum_{\boldsymbol{n}} e^{i\boldsymbol{k}\cdot\boldsymbol{n}}\eta_\mu(n)\left[\bar{\chi}(n)U_\mu(n)\chi(n+\hat{\mu}) + \bar{\chi}(n+\hat{\mu})U_\mu^\dagger(n)\chi(n)\right] \tag{15}$$

# Diagrams of contractions of quark propagators

A correlation function is a contraction of quark propagators generated from sources.




3-point function




2-point function

# Noise U(1) momentum sources and PxP insertion sources

To generate quark with non-zero momentum, we use Noise U(1) momentum sources. In the case of the 3-point functions, we need to insert $(\gamma_5 \otimes \gamma_5)$ operator into quark propagators and make new sources. These sources are called PxP insertion sources.

## Noise U(1) momentum sources

$$h([\boldsymbol{n}, t], b; t'; p) = \delta_{t,t'} \eta(\boldsymbol{n}, b) e^{i\boldsymbol{p}\cdot\boldsymbol{n}} \tag{16}$$

## PxP insertion sources

$$h_2(n, c; t_3; \boldsymbol{k}) = \delta_{t,t_3} \psi_2(n, c; t_1; \boldsymbol{0}) \varepsilon(n) e^{i\boldsymbol{k}\cdot\boldsymbol{n}}; \quad n = (\boldsymbol{n}, t) \tag{17}$$

The quark propagators are obtained by solving the Dirac equation with these sources.

$$\sum_{y,b} [\not{D} + m_f](x, a; y, b) \psi_f(y, b; t'; \boldsymbol{k}) = h(x, a; t'; \boldsymbol{k}) \tag{18}$$

# Double Ratio Method

To determine $f_+(0)$ precisely, we adopt a strategy employed in C. Dawson et al. Phys. Rev., D74:114502, 2006. This is called "double ratio method".

$$
\begin{aligned}
R(t) &= \frac{C_4^{K\pi}(t,t';\mathbf{0},\mathbf{0})\,C_4^{\pi K}(t,t';\mathbf{0},\mathbf{0})}{C_4^{KK}(t,t';\mathbf{0},\mathbf{0})\,C_4^{\pi\pi}(t,t';\mathbf{0},\mathbf{0})} \\
&\xrightarrow[t,(t'-t)\to\infty]{} \frac{\langle\pi(\mathbf{0})|V_4|K(\mathbf{0})\rangle\langle K(\mathbf{0})|V_4|\pi(\mathbf{0})\rangle}{\langle K(\mathbf{0})|V_4|K(\mathbf{0})\rangle\langle\pi(\mathbf{0})|V_4|\pi(\mathbf{0})\rangle} \\
&= \frac{(M_K+M_\pi)^2}{4M_KM_\pi}\left|f_0(q^2_{\mathrm{max}})\right|^2
\end{aligned}
\tag{19}
$$

- $q^2_{max} = (M_K - M_\pi)^2$

# $q^2 = 0$ Interpolation

On the lattice, momenta are quantized so that we cannot obtain $f_0(q^2)$ at point $q^2 = 0$. Thus, we must interpolate the $f_0(q^2)$ data to $q^2 = 0$. There are three kinds of fit.

- Linear fit : $f_0(q^2) = f_0(0)(1 + \lambda_0^{(1)} q^2)$
- quadratic fit : $f_0(q^2) = f_0(0)(1 + \lambda_0^{(1)} q^2 + \lambda_0^{(2)} q^4)$
- pole fit : $f_0(q^2) = f_0(0)/(1 - \lambda_0^{(1)} q^2)$

# Another Method

From the equation of motion,

$$\partial_\mu V^\mu(x) = (m_s - m_u) S(x), \tag{20}$$

where $V^\mu(x) = \bar{s}(x)\gamma^\mu u(x)$ and $m_s(m_u)$ is the mass of the strange(up) quark mass. $S(x)$ is the scalar operator $\bar{s}(x)u(x)$. A hadronic matrix element of $V^\mu(x)$

$$\langle \pi(p')|V^\mu(x)|K(p)\rangle = e^{ix(p'-p)}\langle \pi(p')|V^\mu(0)|K(p)\rangle, \tag{21}$$

from translation invariance.

$$iq_\mu \langle \pi(p')|V^\mu(x)|K(p)\rangle = (m_s - m_u)\langle \pi(p')|S(x)|K(p)\rangle, \tag{22}$$

where $q_\mu = p'_\mu - p_\mu$.

# Form Factors

$$\langle \pi(p')|V^\mu(0)|K(p)\rangle = f_+(q^2)\left[p^\mu + p'^\mu - \frac{m_K^2 - m_\pi^2}{q^2}q^\mu\right] + f_0(q^2)\frac{m_K^2 - m_\pi^2}{q^2}q^\mu, \tag{23}$$

$$\langle \pi(p')|S(0)|K(p)\rangle = \frac{m_K^2 - m_\pi^2}{m_s - m_u} f_0(q^2). \tag{24}$$

Thus, we can calculate the scalar form factor $f_0(q^2)$ from the matrix element of the scalar operator;

$$f_0(q^2) = \frac{m_s - m_u}{m_K^2 - m_\pi^2}\langle \pi(p')|S|K(p)\rangle. \tag{25}$$

Because $f_+(0) = f_0(0)$,

$$f_+(0) = \frac{m_s - m_u}{m_K^2 - m_\pi^2}\langle \pi(p')|S|K(p)\rangle_{q^2=0}. \tag{26}$$

# Scalar Operator

## Scalar Operator

$$S(t,\boldsymbol{k}) = \sum_{\boldsymbol{n},c} \bar{\chi}^c(n)\chi^c(n)e^{i\boldsymbol{k}\cdot\boldsymbol{n}} \tag{27}$$

## Scalar Operator Three-point Functions

$$\begin{aligned} C_I^{K\pi}(t_1,t_2;\boldsymbol{k}_1,\boldsymbol{k}_3) &= \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3} \langle \mathscr{O}_\pi(\boldsymbol{x}_1,t_1+t_0)S(\boldsymbol{x}_2,t_2+t_0)\mathscr{O}_K^\dagger(\boldsymbol{x}_3,t_0)\rangle \\ &\quad \times e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1}e^{i(\boldsymbol{k}_3-\boldsymbol{k}_1)\cdot\boldsymbol{x}_1}e^{i\boldsymbol{k}_3\cdot\boldsymbol{x}} \\ &\xrightarrow[t,(t'-t)\to\infty]{} V_s\frac{Z_K^* Z_\pi}{4E_K(\boldsymbol{k}_3)E_\pi(\boldsymbol{k}_1)} \\ &\quad \times \langle\pi(\boldsymbol{k}_1)|S|K(\boldsymbol{k}_3)\rangle e^{-E_K(\boldsymbol{k}_3)t_2-E_\pi(\boldsymbol{k}_1)(t_1-t_2)} \end{aligned} \tag{28}$$

# Partially twisted boundary condition

Under periodic boundary condition

$$\psi(x + \hat{e}_i L) = \psi(x), \quad i = 1, 2, 3, \tag{29}$$

momenta are quantized as $\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{n}$.
If we twist the boundary condition by $\theta$:

$$\psi(x + \hat{e}_i L) = e^{i\theta_i}\psi(x), \quad i = 1, 2, 3, \tag{30}$$

we can obtain the momenta as

$$p_i = \frac{\theta_i}{L} + \frac{2\pi}{L} n_i, \quad i = 1, 2, 3. \tag{31}$$

# Status of the calculation

- We have implemented the code for this calculation and run it on KISTI supercomputer.
- Data are available on some gauge configurations, but we are checking the validity of the data.
- We will check the gauge invariance of the data by transforming the gauge configurations with random SU(3) matrices, $g(x) \in SU(3)$.

$$U(x, x+\hat{\mu}) \to g(x) U(x, x+\hat{\mu}) g^{\dagger}(x+\hat{\mu}). \tag{32}$$

- We will also check the Ward identity,

$$\partial_\mu V^\mu(x) = (m_s - m_u) S(x). \tag{33}$$