Determining K_{l3} Decay Form Factors at Zero Momentum Transfer

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The Standard Model of Particle Physics

There are FOUR FORCES in nature:

- Electromagnetic Force
- Strong Nuclear Force
- Weak Nuclear Force
- Gravitational Force

The Electroweak Theory

- Unify the electromagnetism and the weak interaction to $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge theory.
- Yukawa interaction terms give fermions their masses when the SU(2) gauge symmetry is spontaneously broken.
- Diagonalizing this terms by bi-unitary matrices results in quark-mixing charged weak interactions.

Cabibbo-Kobayashi-Maskawa Matrix

CKM Matrix

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{1}$$

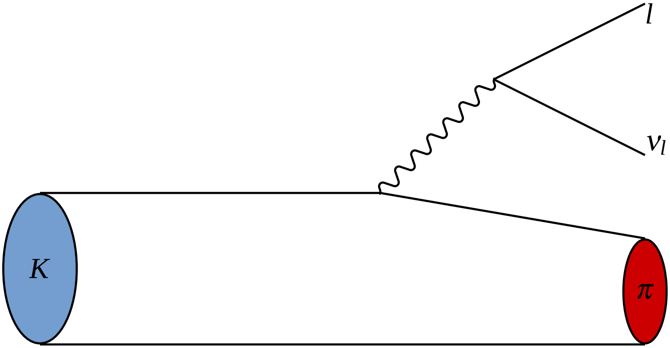
Unitarity on the first row

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \tag{2}$$

V_{us}

- To test the unitarity, it is most important to determine V_{us} accurately.
- The reason is that V_{ub} is negligibly small, and V_{ud} is determined in high precision.
- Precise determination of V_{us} is also important in the Wolfenstein parameterization. (CP violation)
- $|V_{us}|$ is traditionally obtained from the experimental rate for K_{l3} decays.

K_{l3} Decays



$$\Gamma(K_{l3}) = \frac{G_F^2}{192\pi^3} M_K^5 C^2 I |V_{us}|^2 |f_+(0)|^2 S_{\text{ew}} (1 + 2\Delta_{\text{SU}(2)} + 2\Delta_{\text{em}}) \tag{3}$$

Form Factors

$$\langle \pi(p')|V_\mu|K(p)\rangle = C\{(p_\mu + p'_\mu)f_+(q^2) + (p_\mu - p'_\mu)f_-(q^2)\} \quad (4)$$

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2) \quad (5)$$

- $q^2 = (p - p')^2$.
- $V_\mu = \bar{s}\gamma_\mu u$.
- By construction, $f_0(0) = f_+(0)$.
- In SU(3) limit, $f_+(0) = 1$.
- Need non-perturbative calculation.

Lattice QCD

Euclidean Path Integral

$$\langle \mathscr{O} \rangle = \frac{1}{Z} \int \mathscr{D}[U]\mathscr{D}[\bar{\psi}]\mathscr{D}[\psi]\mathscr{O}[U,\psi,\bar{\psi}]e^{-S_G[U]-S_F[U,\psi,\bar{\psi}]}$$
$$= \frac{1}{Z} \int \mathscr{D}[U]e^{-S_{\text{eff}}[U]}A[U] \sum_{z_1\cdots z_n} \varepsilon^{z_1 z_2 \cdots z_n}_{y_1 y_2 \cdots y_n} M[U]^{-1}_{z_1 x_1} M[U]^{-1}_{z_2 x_2} \cdots M[U]^{-1}_{z_n x_n} \quad (6)$$

$$S_F = \bar{\psi} M[U] \psi \quad (7)$$

$$\mathscr{O}[U,\psi,\bar{\psi}] = \psi_{y_1}\bar{\psi}_{x_1}\psi_{y_2}\bar{\psi}_{x_2}\cdots\psi_{y_n}\bar{\psi}_{x_n}A[U] \quad (8)$$

$$S_{\text{eff}} = S_G[U] - \ln\det M[U] = S_G[U] - \mathrm{Tr}\ln M[U] \quad (9)$$

$$\langle \mathscr{O} \rangle = \lim_{T\to\infty} \frac{1}{Z_T} \mathrm{Tr}\left[e^{-HT}\mathscr{O}\right], \quad (10)$$

where $Z_T = \mathrm{Tr}[e^{-HT}]$.

Three-point Correlation Functions

$$
\begin{aligned}
C_{\mu}^{PQ}(t_1, t_2; \vec{k}_1, \vec{k}_3) &= \sum_{\vec{x}_1, \vec{x}_2, \vec{x}_3} \langle \mathscr{O}_Q(\vec{x}_1, t_1 + t_0) V_\mu(\vec{x}_2, t_2 + t_0) \mathscr{O}_P^\dagger(\vec{x}_3, t_0) \rangle \\
&\quad \times e^{i\vec{k}_1 \cdot \vec{x}_1} e^{i(\vec{k}_3 - \vec{k}_1)\cdot \vec{x}_2} e^{-i\vec{k}_3 \cdot \vec{x}_3} \\
&\xrightarrow[t_2, (t_1 - t_2) \to \infty]{} V_s \frac{Z_P^* Z_Q}{4 E_Q(\vec{k}_1) E_P(\vec{k}_3)} \langle Q(\vec{k}_1) | V_\mu | P(\vec{k}_3) \rangle \\
&\quad \times e^{-E_Q(\vec{k}_1)(t_1 - t_2)} e^{-E_P(\vec{k}_3) t_2}.
\end{aligned} \tag{11}
$$

Two-point Correlation Functions

$$
\begin{aligned}
C^P(t;\vec{k}_1) &= \sum_{\vec{x}_1,\vec{x}_2} \langle \mathcal{O}_P(\vec{x}_1, t+t_0)\,\mathcal{O}_P^\dagger(\vec{x}_2, t_0)\rangle e^{i\vec{k}_1\cdot(\vec{x}_1-\vec{x}_2)} \\
&\xrightarrow[t\to\infty]{} V_s \frac{Z_P^* Z_P}{2E_P(\vec{k}_1)} e^{-E_P(\vec{k}_1)t}
\end{aligned}
\tag{12}
$$

Interpolating Operators

Staggered Fermion Action

$$S_F = a^4 \sum_n \left\{ \sum_\mu \frac{1}{2a} \eta_\mu(n) \left[\bar{\chi}(n) U_\mu(n) \chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu}) U_\mu^\dagger(n) \chi(n) \right] + m\bar{\chi}(n)\chi(n) \right\}, \tag{13}$$

where $\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu}$.

Pseudo-scalar Meson Operator

$$P(t,\vec{k}) = \sum_{\vec{n}} \varepsilon(n) \bar{\chi}(n) \chi(n) e^{i\vec{k}\cdot\vec{n}}, \quad \varepsilon(n) = (-1)^{\sum_\nu n_\nu} \tag{14}$$

Vector Current Operator

$$V_\mu(t,\vec{k}) = \frac{1}{2} \sum_{\vec{n}} e^{i\vec{k}\cdot\vec{n}} \eta_\mu(n) \left[\bar{\chi}(n) U_\mu(n) \chi(n+\hat{\mu}) + \bar{\chi}(n+\hat{\mu}) U_\mu^\dagger(n) \chi(n) \right] \tag{15}$$

Three-point Correlator

$$
\begin{aligned}
&\langle P_\pi(t_1,\vec{k}_1)V_\mu(t_2,\vec{k}_2)P_K(t_3,\vec{k}_3)\rangle \\
&\quad = \Big\langle \sum_{\vec{n}_1,c} \varepsilon(n_1)\bar{\chi}_1^c(n_1)\chi_2^c(n_1)e^{i\vec{k}_1\cdot\vec{n}_1} \\
&\qquad \frac{1}{2}\sum_{\vec{n}_2,a,b} e^{i\vec{k}_2\cdot\vec{n}_2}\eta_\mu(n_2)[\bar{\chi}_3^a(n_2)U_\mu^{ab}(n_2)\chi_1^b(n_2+\hat{\mu}) \\
&\qquad\qquad + \bar{\chi}_3^a(n_2+\hat{\mu})U_\mu^{\dagger ab}(n_2)\chi_1^b(n)] \\
&\qquad \sum_{\vec{n}_3,d} \varepsilon(n_3)\bar{\chi}_2^d(n_3)\chi_3^d(n_3)e^{i\vec{k}_3\cdot\vec{n}_3}\Big\rangle
\end{aligned} \tag{16}
$$

Wick Contraction

$$-\frac{1}{2}\sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3}\sum_{a,b,c,d} e^{i\vec{k}_1\cdot\vec{n}_1+i\vec{k}_2\cdot\vec{n}_2+i\vec{k}_3\cdot\vec{n}_3}\varepsilon(n_1)\varepsilon(n_3)\eta_\mu(n_2)$$
$$[U_\mu^{ab}(n_2)G_1(n_2+\hat{\mu},b;n_1,c)G_2(n_1,c;n_3,d)G_3(n_3,d;n_2,a)$$
$$+U_\mu^{\dagger ab}(n_2)G_1(n_2,b;n_1,c)G_2(n_1,c;n_3,d)G_3(n_3,d;n_2+\hat{\mu},a)] \tag{17}$$

- Green's functions are inverse matrices of fermion Dirac matrices.
- Too many computations!

Wall Sources Method

Noise U(1) Vectors

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\eta} \eta(\vec{n}, c)\eta^*(\vec{n}', c') = \delta_{\vec{n},\vec{n}'}\delta_{c,c'} \tag{18}$$

Noise U(1) Sources

$$h_1(n, b; t_1; \vec{k}) = \delta_{t,t_1}\eta(\vec{n}, b)e^{i\vec{k}\cdot\vec{n}}; \quad n = (\vec{n}, t). \tag{19}$$

Dirac Equations

$$\sum_{y,b} [\not{D} + m_1](x, a; y, b)\psi_1(y, b; t_1; \vec{k}_1) = h_1(x, a; t_1; \vec{k}_1), \tag{20}$$

$$\sum_{y,b} [\not{D} + m_2](x, a; y, b)\psi_2(y, b; t_1; \vec{0}) = h_1(x, a; t_1; \vec{0}). \tag{21}$$

PxP Insertion Sources

$$
\begin{aligned}
&-\frac{1}{2N}\sum_{\eta}\sum_{\vec{n}_2,\vec{n}_3}\sum_{a,b,d} e^{i\vec{k}_2\cdot\vec{n}_2+i\vec{k}_3\cdot\vec{n}_3}\eta_\mu(n_2) \\
&[U_\mu^{ab}(n_2)\psi_1(n_2+\hat{\mu},b;t_1;\vec{k}_1)\psi_2^*(n_3,d;t_1;\vec{0})G_3(n_3,d;n_2,a) \\
&+U_\mu^{\dagger ab}(n_2)\psi_1(n_2,b;t_1;\vec{k}_1)\psi_2^*(n_3,d;t_1;\vec{0})G_3(n_3,d;n_2+\hat{\mu},a)],
\end{aligned} \tag{22}
$$

γ_5-hermiticity of the Dirac operator

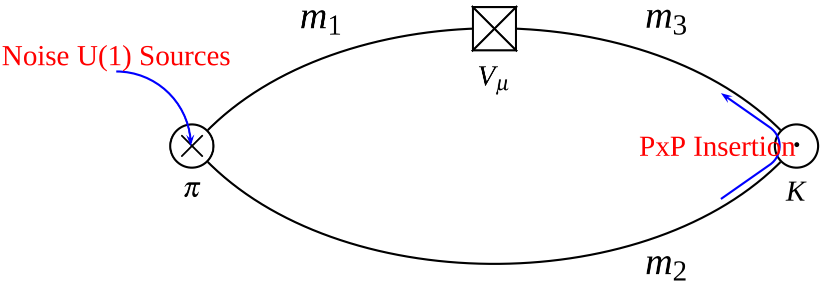
$$
G^*(x,a;y,b) = \varepsilon(x)\varepsilon(y)G(y,b;x,a) \tag{23}
$$

PxP Insertion Sources

$$
h_2(n,c;t_3;\vec{k}) = \delta_{t,t_3}\psi_2(n,c;t_1;\vec{0})\varepsilon(n)e^{i\vec{k}\cdot\vec{n}}; \quad n = (\vec{n},t), \tag{24}
$$

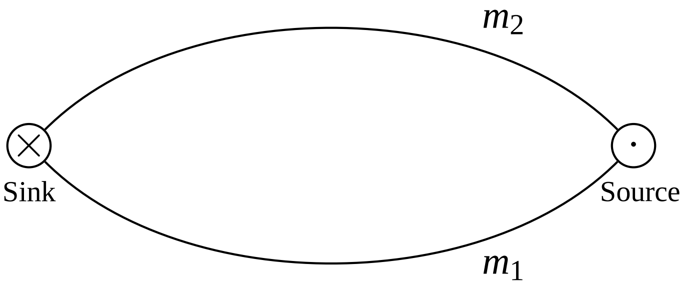
Final Formula

$$
\begin{aligned}
& -\frac{1}{N}\frac{1}{2}\sum_{\eta}\sum_{\vec{n}_2}\varepsilon(n_2)\eta_\mu(n_2)e^{i\vec{k}_2\cdot\vec{n}_2} \\
& \quad [\tilde{\psi}^\dagger_{32}(n_2;t_3;-\vec{k}_3)U_\mu(n_2)\psi_1(n_2+\hat{\mu};t_1;\vec{k}_1) \\
& \quad -\tilde{\psi}^\dagger_{32}(n_2+\hat{\mu};t_3;-\vec{k}_3)U^\dagger_\mu(n_2)\psi_1(n_2;t_1;\vec{k}_1)].
\end{aligned}
\tag{25}
$$



Two-point Correlator

$$\langle P(t,\vec{k})P(t',-\vec{k})\rangle = -\frac{1}{N}\sum_{\eta}\sum_{\vec{n}} e^{i\vec{k}\cdot\vec{n}} \sum_{c} \psi_1(n,c;t';\vec{0})\,\psi_2^*(n,c;t';\vec{k}) \quad (26)$$

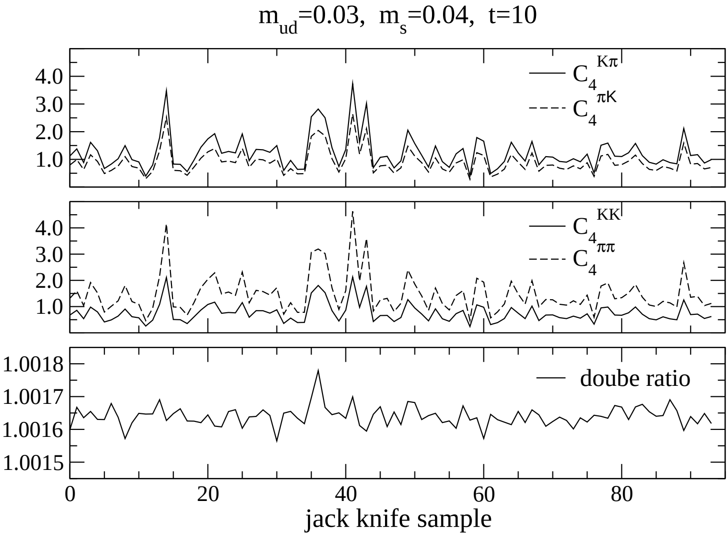


Double Ratio Method

$$
\begin{aligned}
R(t) &= \frac{C_4^{K\pi}(t,t';\vec{0},\vec{0})C_4^{\pi K}(t,t';\vec{0},\vec{0})}{C_4^{KK}(t,t';\vec{0},\vec{0})C_4^{\pi\pi}(t,t';\vec{0},\vec{0})} \\
&\xrightarrow[t,(t'-t)\to\infty]{} \frac{\langle\pi(\vec{0})|V_4|K(\vec{0})\rangle\langle K(\vec{0})|V_4|\pi(\vec{0})\rangle}{\langle K(\vec{0})|V_4|K(\vec{0})\rangle\langle\pi(\vec{0})|V_4|\pi(\vec{0})\rangle} \\
&= \frac{(M_K+M_\pi)^2}{4M_KM_\pi}\left|f_0(q^2_{\rm max})\right|^2
\end{aligned}
\tag{27}
$$

- $q^2_{max} = (M_K - M_\pi)^2$

C. Dawson *et al.* Phys. Rev., D74:114502, 2006



Double Ratio Method

$$\begin{aligned}\tilde{R}(t;\vec{p},\vec{p}') &= \frac{C_4^{K\pi}(t,t';\vec{p},\vec{p}')C^K(t;\vec{0})C^\pi(t'-t;\vec{0})}{C_4^{K\pi}(t,t';\vec{0},\vec{0}')C^K(t;\vec{p})C^\pi(t'-t;\vec{p}')} \\ &\xrightarrow[t,(t'-t)\to\infty]{} \frac{\langle\pi(\vec{p}')|V_4|K(\vec{p})\rangle}{\langle\pi(\vec{0})|V_4|K(\vec{0})\rangle} \\ &= \frac{E_K(\vec{p})+E_\pi(\vec{p}')}{M_K+M_\pi}F(\vec{p},\vec{p}'),\end{aligned} \tag{28}$$

where

$$F(\vec{p},\vec{p}') = \frac{f_+(q^2)}{f_0(q^2_{\max})}\left(1+\frac{E_K(\vec{p})-E_\pi(\vec{p}')}{E_K(\vec{p})+E_\pi(\vec{p}')}\xi(q^2)\right) \tag{29}$$

$$\xi(q^2) = \frac{f_-(q^2)}{f_+(q^2)} \tag{30}$$

Double Ratio Method

$$R_i(t;\vec{p},\vec{p}') = \frac{C_i^{K\pi}(t,t';\vec{p},\vec{p}')C_4^{KK}(t,t';\vec{p},\vec{p}')}{C_4^{K\pi}(t,t';\vec{p},\vec{p}')C_i^{KK}(t,t';\vec{p},\vec{p}')}, \quad (i=1,2,3) \tag{31}$$

Using R_i, we can extract $\xi(q^2)$;

$$\begin{aligned} &\frac{-(E_K(\vec{p})+E_K(\vec{p}'))(p+p')_i + (E_k(\vec{p})+E_\pi(\vec{p}'))(p+p')_i R_i}{(E_K(\vec{p})+E_K(\vec{p}'))(p-p')_i - (E_k(\vec{p})-E_\pi(\vec{p}'))(p+p')_i R_i} \\ &\xrightarrow[t,(t'-t)\to\infty]{} \xi(q^2) \end{aligned} \tag{32}$$

$q^2 = 0$ Interpolation

Model-dependent fit

- Linear fit : $f_0(q^2) = f_0(0)(1 + \lambda_0^{(1)} q^2)$
- quadratic fit : $f_0(q^2) = f_0(0)(1 + \lambda_0^{(1)} q^2 + \lambda_0^{(2)} q^4)$
- pole fit : $f_0(q^2) = f_0(0)/(1 - \lambda_0^{(1)} q^2)$

Chiral Extrapolation

We simulate at non-physical quark masses, so we need to extrapolate to physical point.

Fitting $f_+(0)$ versus M_K^2 and M_π^2 using Chiral Perturbation Theory.

Another Method

From the equation of motion,

$$\partial_\mu V^\mu(x) = (m_s - m_u)S(x), \tag{33}$$

where $V^\mu(x) = \bar{s}(x)\gamma^\mu u(x)$ and $m_s(m_u)$ is the mass of the strange(up) quark mass. $S(x)$ is the scalar operator $\bar{s}(x)u(x)$. A hadronic matrix element of $V^\mu(x)$

$$\langle \pi(p')|V^\mu(x)|K(p)\rangle = e^{ix(p'-p)}\langle \pi(p')|V^\mu(0)|K(p)\rangle, \tag{34}$$

from translation invariance.

$$q_\mu \langle \pi(p')|V^\mu(x)|K(p)\rangle = (m_s - m_u)\langle \pi(p')|S(x)|K(p)\rangle, \tag{35}$$

where $q_\mu = p'_\mu - p_\mu$.

Form Factors

$$\langle\pi(p')|V^{\mu}(0)|K(p)\rangle = f_{+}(q^2)\left[p^{\mu}+p'^{\mu}-\frac{m_K^2-m_\pi^2}{q^2}q^{\mu}\right]+f_0(q^2)\frac{m_K^2-m_\pi^2}{q^2}q^{\mu}, \tag{36}$$

$$\langle\pi(p')|S(0)|K(p)\rangle = \frac{m_K^2-m_\pi^2}{m_s-m_u}f_0(q^2). \tag{37}$$

Thus, we can calculate the scalar form factor $f_0(q^2)$ from the matrix element of the scalar operator;

$$f_0(q^2) = \frac{m_s-m_u}{m_K^2-m_\pi^2}\langle\pi(p')|S|K(p)\rangle. \tag{38}$$

Because $f_{+}(0) = f_0(0)$,

$$f_{+}(0) = \frac{m_s-m_u}{m_K^2-m_\pi^2}\langle\pi(p')|S|K(p)\rangle_{q^2=0}. \tag{39}$$

Scalar Operator

$$S(t,\vec{k}) = \sum_{\vec{n},c} \bar{\chi}^c(n)\chi^c(n)e^{i\vec{k}\cdot\vec{n}}. \tag{40}$$

Scalar Operator Three-point Functions

$$\begin{aligned} C_I^{K\pi}(t_1,t_2;\vec{k}_1,\vec{k}_3) &= \sum_{\vec{x}_1,\vec{x}_2,\vec{x}_3} \langle \mathscr{O}_\pi(\vec{x}_1,t_1+t_0)S(\vec{x}_2,t_2+t_0)\mathscr{O}_K^\dagger(\vec{x}_3,t_0)\rangle \\ &\quad \times e^{i\vec{k}_1\cdot\vec{x}_1}e^{i(\vec{k}_3-\vec{k}_1)\cdot\vec{x}_1}e^{i\vec{k}_3\cdot\vec{x}} \\ &\xrightarrow[t,(t'-t)\to\infty]{} V_s \frac{Z_K^* Z_\pi}{4E_K(\vec{k}_3)E_\pi(\vec{k}_1)} \\ &\quad \times \langle \pi(\vec{k}_1)|S|K(\vec{k}_3)\rangle e^{-E_K(\vec{k}_3)t_2 - E_\pi(\vec{k}_1)(t_1-t_2)}. \end{aligned} \tag{41}$$

Three-point Correlator

$$-\frac{1}{N}\frac{1}{2}\sum_{\eta}\sum_{\vec{n}_2}\varepsilon(n_2)e^{i\vec{k}_2\cdot\vec{n}_2}\tilde{\psi}^{\dagger}_{32}(n_2;t_3;-\vec{k}_3)\psi_1(n_2;t_1;\vec{k}_1), \tag{42}$$

Future Plan

- Twisted boundary condition
- Staggered chiral perturbation theory